Method of moments and collective motion of nuclei

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Studies in which a new method for describing the collective motion of nuclei based on the phase-space moments of the Wigner function was proposed and developed are reviewed. The results of calculations made in the framework of this method are given for the equilibrium shapes of rotating nuclei, the energies and excitation probabilities of isoscalar and isovector resonances and low-lying 0^+, 1^+, 2^+, 3^+, 4^+ and 1^-, 2^-, 3^- collective states, and their widths. These results are discussed and compared with experimental data and calculations of other authors.

INTRODUCTION

The rapid development of experimental nuclear physics that has occurred during the last two decades and still continues has led to the accumulation of information qualitatively different from that which existed at the time when the basic models of nuclear structure were formulated.

The accuracy with which the properties of the ground and low-lying nuclear states are measured has been increased. Data on the decay properties of states are being steadily accumulated and made more accurate. The ranges of excitation energies and angular momenta in which new data are published have been very greatly extended. There has been a remarkable increase in the number of nuclear states for which information that determines their electromagnetic properties is accessible.

The main bulk of the new experimental information that directly relates to nuclear structure comes at the present time from experiments that use nuclear reactions, in particular ones with heavy ions. Analysis of nuclear reactions requires certain information about nuclear structure. Thus, there is an ever increasing need for joint analysis of the dynamics of nuclear reactions and of the structure of the stationary states of nuclei. For reliable analysis of both reactions and structure it is necessary to develop a theory in which a balance is achieved in the accuracy with which the basic structural features of the nuclei and the reaction mechanism are described. This aim can be achieved only by the formulation of nuclear models that operate basically with reliably established nuclear parameters and do not require excessively complicated calculations to obtain quantitative relations. At the same time, the model should not be purely phenomenological but should, if possible, be derived from "first principles," for example, from a many-particle Schrödinger equation. The original approach to the investigation of collective motion considered in this review satisfies in general and in the main these criteria. It is based on a method that has performed well in the theory of atoms, the Hartree–Fock method,[1,2] which can also be regarded as the best suited for the study of nuclei. For the description of excited states it was generalized by Dirac, who proposed the time-dependent Hartree–Fock (TDHF) method. This method serves explicitly or indirectly as the foundation of many phenomenological, microscopic, and mixed nuclear models.[3] From the TDHF method one can derive the currently very popular random-phase approximation (RPA), which describes small-amplitude vibrational motions (small vibrations about an equilibrium state).[2] There have also been attempts to study large-amplitude motions by the method.[4]

A powerful stimulus to the development of the "hydrodynamic" aspects of the TDHF method was the discovery of giant resonances of multipolarity $\lambda \geqslant 2$. The fact is that the giant dipole resonance (GDR) (and also the monopole resonance—the breathing mode) cannot be described well in a purely hydrodynamic approach,[5] for example, the well-known Goldhaber–Teller and Steinwedel–Jensen models. For the energy of the isoscalar giant quadrupole resonance (GQR) the hydrodynamic approach was not even able to give the order of magnitude.

A connection between the TDHF method and hydrodynamics can be established as follows.[6] First, by a Fourier transformation with respect to the coordinate $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, the density matrix $\rho(\mathbf{r}_1,\mathbf{r}_2)$ that appears in the method is transformed into the Wigner function (Ref. 7) $f(\mathbf{R}, \mathbf{p})$, where $\mathbf{R} = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2)$. In its form, the equation obtained for $f(\mathbf{R}, \mathbf{p})$ is very similar to the Vlasov kinetic equation,[8] differing from it by quantum corrections that generally do not play an important part. This equation is then integrated over the momenta with weights 1 and p_i, giving a continuity equation and an equation of motion (Euler equation). The dynamical variables in them are the density $\rho(\mathbf{r})$ and the mean matter velocity $\mathbf{u}(\mathbf{r})$. The Euler equation also contains one further dynamical variable—the pressure tensor $P_{ij}(\mathbf{r})$. For it, one can also write down a dynamical equation by integrating the "kinetic" equation with weight $p_i p_j$. However, this step leads to a dead end, since each new integration necessarily adds a new dynamical variable, so that ultimately one obtains an infinite system of coupled differential equations of hydrodynamic type. To obtain a closed system of equations for the variables $\rho(\mathbf{r})$ and $\mathbf{u}(\mathbf{r})$, one can attempt to represent the pressure tensor as an approximate function of these local dynamical variables.[9] In many-particle systems in which the mean free path is appreciably shorter than the characteristic distances over which the density varies such a procedure works well. It is usually realized by a local equation of state, which for a degenerate Fermi system can be obtained

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from the Thomas–Fermi approximation with density-dependent potentials. In nuclei, the mean free path is greater than the nuclear diameter, so that for them such a prescription is obviously invalid.[10]

Bertsch[11] was the first who proposed that a nucleus could vibrate as an elastic body, a characteristic property of which is the presence of nondiagonal components of the pressure tensor, which are absent in the hydrodynamic picture. The presence of these components can be explained microscopically by a typically quantum effect inherent in Fermi systems—deformation of the Fermi surface in the momentum space during nuclear motion. Bertsch proposed that this effect should be taken into account macroscopically, by means of "scaling." The basis of the method is a simple idea, namely, a collective state of the system, if it exists, is characterized by the property that all the single-particle states forming the Slater determinant have the same displacement vector and velocity potential. Bertsch's pioneering paper, which gave entirely reasonable values for the energies of the giant monopole and quadrupole resonances, was followed by many studies that improved, justified, and generalized the scaling approximation (see the references in Sec. 1). Only isolated publications were devoted to the "dead-end" path of development of the "hydrodynamic" approach. For example, Nix and Sierk[12] added to the continuity and Euler equations a dynamical equation for the pressure tensor, from which the third-rank tensor was thrown out, i.e., the problem of closing the infinite hierarchy of equations was "solved" in butcher's fashion. Nevertheless, Nix and Sierk obtained unexpectedly good agreement with experiment for the GQR energy. Their method of calculating the energies was rather complicated, so that they were also forced to ignore all potential interactions between nucleons.

To overcome these basically technical difficulties, the authors of Refs. 13–15 used the method of virial theorems that had been developed by Chandrasekhar and Lebovitz[16] to solve similar problems in astronomy—the calculation of figures of equilibrium and the eigenfrequencies of the vibrations of rotating self-gravitating masses. This method not only made it possible to calculate the GQR energy with allowance for all potential interactions and generalize the result to rotating nuclei but also to solve the most important problem, namely, the closure problem.

Of course, the advantages of the virial-theorem method are not restricted to this. It became clear quite soon that, in conjunction with equations of hydrodynamic type, the virial theorems are a powerful tool for studying collective motion in nuclei and, in general, in all finite many-particle systems whose dynamics can be described by the TDHF equations. Accordingly, this review is mainly devoted to a description of the method of closing the infinite hierarchy of equations and to the applications of the method to the study of the vibrations of rotating and nonrotating nuclei.

1. FORMULATION OF THE METHOD

Equations for the density matrix and Wigner function

The basis of our method of describing collective nuclear dynamics is the equation for the single-particle density matrix $\hat{\rho} = \rho(\mathbf{r}_1,\mathbf{r}_2,t)$:

$$i\hbar\partial\hat{\rho}/\partial t=[\hat{H},\hat{\rho}], \tag{1}$$

where $\hat{H}$ is the self-consistent single-particle Hamiltonian, which, in its turn, depends on the density matrix. Equation (1), with a completely specific definition of the Hamiltonian, appears in Hartree–Fock theory; it is also used in the so-called energy-functional method, which leaves wide possibilities for the choice of the single-particle Hamiltonian and, in addition, gives some grounds for believing that Eq. (1) is fairly general.[17] To study rotating nuclei, it is sufficient to add to the self-consistent Hamiltonian the term $-\mathbf{I}\cdot\boldsymbol{\Omega}$, where $\mathbf{I}$ is the angular-momentum operator, and $\boldsymbol{\Omega}$ is the angular velocity. The influence of external fields can be taken into account by adding a potential of the external forces to $\hat{H}$.

Finally, adding to the right-hand side of the equation the so-called correlation term, one can arrive at an exact equation for the single-particle density matrix, this being the first in the hierarchy of equations for the many-particle density matrices.[18,19]

For the exposition that follows, it is convenient to reformulate Eq. (1), introducing a Wigner transformation of the density matrix:[7]

$$f(\mathbf{r},\mathbf{p},t)=\frac{1}{(2\pi\hbar)^3}\int e^{-i\mathbf{p}\mathbf{s}/\hbar}\rho\left(\mathbf{r}+\frac{\mathbf{s}}{2},\mathbf{r}-\frac{\mathbf{s}}{2},t\right)d\mathbf{s}. \tag{2}$$

The Wigner function $f(\mathbf{r},\mathbf{p},t)$ resembles the distribution function of classical kinetic theory, which gives the probability of finding a nucleon with momentum $\mathbf{p}$ at the point $\mathbf{r}$ at the time t.[19] However, in contrast to a true distribution function, it can, in principle, take negative values.

The Wigner transform of the operators is normalized as follows:[4]

$$O_W(\mathbf{r},\mathbf{p})=\int e^{-i\mathbf{p}\mathbf{s}/\hbar}\left\langle\mathbf{r}+\frac{\mathbf{s}}{2}\middle|\hat{O}\middle|\mathbf{r}-\frac{\mathbf{s}}{2}\right\rangle d\mathbf{s}. \tag{3}$$

Some important properties of the Wigner transformation are given in Ref. 20. Any operator function that depends only on $\hat{p}$ or only on $\hat{r}$ is transformed into the same function of the corresponding variables ($\mathbf{p}$ or $\mathbf{r}$). A product of two operators, $\hat{c}=\hat{a}\cdot\hat{b}$, is transformed to the form

$$c_W(\mathbf{r},\mathbf{p})=\exp\left\{\frac{i\hbar}{2}(\nabla_{\mathbf{r}}^a\cdot\nabla_{\mathbf{p}}^b-\nabla_{\mathbf{p}}^a\cdot\nabla_{\mathbf{r}}^b)\right\}a_W(\mathbf{r},\mathbf{p})b_W(\mathbf{r},\mathbf{p}). \tag{4}$$

The superscript of the nabla operator indicates the function to which the given operator is applied. Applying (4) to (1), we find[20–23]

$$\frac{\partial f}{\partial t}=\frac{2}{\hbar}\sin\left\{\frac{\hbar}{2}(\nabla_{\mathbf{r}}^{\mathrm{H}}\cdot\nabla_{\mathbf{p}}^f-\nabla_{\mathbf{p}}^{\mathrm{H}}\cdot\nabla_{\mathbf{r}}^f)\right\}H_W f. \tag{5}$$

We consider a Hamiltonian that is the sum of a kinetic energy and a local potential. (Nonlocal terms can be readily introduced if needed.) Its Wigner transform is the classical version of the same Hamiltonian:

$$H_W=\frac{p^2}{2m}+W(\mathbf{r}),$$

where W is the potential of the nuclear and Coulomb forces, and m is the nucleon mass. Then from Eq. (5) we have

$$\frac{\partial f}{\partial t}+\frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{r}}f=\frac{2}{\hbar}\sin\left(\frac{\hbar}{2}\nabla_{\mathbf{r}}^{W}\cdot\nabla_{\mathbf{p}}^{f}\right)W(\mathbf{r})f(\mathbf{r},\mathbf{p},t). \tag{6}$$

The approximation in which only the first term of the series expansion of the sine is retained is known as the semiclassical approximation of Hartree–Fock theory. Then Eq. (6) is identical to the kinetic equation for the distribution function (Vlasov equation[6,24]) and provides the basis of many nuclear models (Refs. 6, 12, 20, 23, 25, and 26). The method considered here makes it possible to take into account any number of terms of the expansion in Eq. (6) and, thus, to investigate the part played by the quantum corrections to the solutions of the kinetic equation.[27,28]

Moments with respect to the momenta

The next step in the description of collective nuclear properties is to consider the moments of the Wigner function:

$$n(\mathbf{r},t)=\int f(\mathbf{r},\mathbf{p},t)d\mathbf{p},$$

$$\rho(\mathbf{r},t)=mn(\mathbf{r},t),$$

$$\mathbf{u}(\mathbf{r},t)=\int \mathbf{p}f(\mathbf{r},\mathbf{p},t)d\mathbf{p}/\rho(\mathbf{r},t),$$

$$P_{ij}=\frac{1}{m}\int (p_i-mu_i)(p_j-mu_j)f(\mathbf{r},\mathbf{p},t)d\mathbf{p},$$

$$P_{i_1\ldots i_n}=m^{1-n}\int (p_{i_1}-mu_{i_1})(p_{i_2}-mu_{i_2})\ldots(p_{i_n}-mu_{i_n})f(\mathbf{r},\mathbf{p},t)d\mathbf{p}. \tag{7}$$

The physical meaning of the first moments is obvious: n is the particle number density (ρ is the mass density); $\mathbf{u}$ is the field of the collective velocities; P_{ij} are the components of the pressure tensor (stresses); and P_{ijk} is the tensor of the energy and momentum transport. The need to introduce other tensors will become clear below.

We write down equations of motion for all these moments. A prescription for obtaining them is already suggested by the superficial similarity between the purely quantum equation for the Wigner function and the classical kinetic equation. Therefore, we shall proceed exactly as is done when hydrodynamic equations are derived from a kinetic equation.[24,29,30]

We integrate Eq. (6) over the momenta with weights 1, p_i, and p_ip_j/m. Thus, we obtain the continuity equation

$$\partial n/\partial t+\operatorname{div}(n\mathbf{u})=0, \tag{8}$$

the equation of motion (generalization of the well-known Euler equation)

$$\frac{\partial}{\partial t}(\rho u_i)+n\frac{\partial W}{\partial x_i}+\sum_{j=1}^{3}\frac{\partial}{\partial x_j}(P_{ij}+\rho u_iu_j)=0, \tag{9}$$

and an equation for the pressure tensor:

$$\frac{\partial P_{ij}}{\partial t}+\sum_{k=1}^{3}\left(P_{jk}\frac{\partial u_i}{\partial x_k}+P_{ik}\frac{\partial u_j}{\partial x_k}\right)+\sum_{k=1}^{3}\frac{\partial}{\partial x_k}(P_{ijk}+P_{ij}u_k)=0. \tag{10}$$

These three equations do not in any way differ from the classical equations of hydrodynamics, though it is true that for them the last equation is usually replaced by its trace, the energy balance equation. A quantum correction first appears in the equation for the third-rank tensor P_{ijk}. The equation of motion for a tensor of arbitrary rank n is obviously obtained by integrating (6) with weight $p_{i_1}\ldots p_{i_n}/m^{n-1}$:

$$\frac{\partial}{\partial t}P_{i_1\ldots i_n}+\sum_{k=1}^{3}\frac{\partial}{\partial x_k}(P_{i_1\ldots i_nk}+P_{i_1\ldots i_n}u_k)+\sum_{l=1}^{n}\sum_{k=1}^{3}\left\{\frac{\partial u_{i_l}}{\partial x_k}P_{i_1\ldots(i_l)\ldots i_nk}-\frac{1}{\rho}\frac{\partial P_{i_lk}}{\partial x_k}P_{i_1\ldots(i_l)\ldots i_n}\right\}$$
$$=-\sum_{k=1}^{\left[\frac{n-1}{2}\right]}\frac{(-1)^k n!(\hbar/2m)^{2k}/m}{((2k+1)!)^2(n-2k-1)!}\times\sum_{P\binom{i_1\ldots i_n}{l_1\ldots l_n}}P_{(l_1\ldots l_{2k+1})l_{k+2}\ldots l_n}\frac{\partial^{2k+1}W}{\partial x_{l_1}\ldots\partial x_{l_{2k+1}}}. \tag{11}$$

Here, $P_{(l_1\ldots l_m)l_{m+1}\ldots l_n}$ denotes a tensor with $n-m$ indices, which remain after deletion of the indices $l_1\ldots l_m$ from the set $i_1\ldots i_n$,

$$P_{(l_1l_2\ldots l_n)}=\rho,\quad P_{(l_1l_2\ldots l_{n-1})l_n}=0.$$

The summation on the right-hand side of Eq. (11) is over k and over all permutations of the indices $i_1\ldots i_n$, while $[(n-1)/2]$ denotes the integer part of the number $\frac{1}{2}(n-1)$.

For the Wigner function the natural boundary condition is $f\to 0$ as $|\mathbf{p}|\to\infty$. Obviously, for finite nuclei we must also adopt a second boundary condition $f\to 0$ as $|\mathbf{r}|\to\infty$, in order to ensure that the various nuclear variables defined in (7) (n, $\rho\mathbf{u}$, P_{ij}, etc.) are nonvanishing only within the nucleus.

The number of equations for the moments of the Wigner function is infinite. It is readily seen that they form a hierarchy—the time derivative of the density is related to the velocity of the nuclear matter, whose time derivative is, in its turn, related to the pressure tensor, etc.

The term that "couples" the equation for the tensor $P_{i_1\ldots i_n}$ to the next equation in the chain is the tensor $P_{i_1\ldots i_ni_{n+1}}$, which has in the integrand an additional velocity

factor $(\mathbf{p} - m\mathbf{u})_{i_{n+1}}$. Practically all physical applications of the theory of collective nuclear motion are restricted by the following condition: The energy of the zero-point motion of the nucleons [$(3/5)T_F A \simeq 21A$ MeV] is much greater than the kinetic energy of the collective flow, i.e., in typical cases $|p_i| \gg m|u_i|$ and, therefore, the coupling term does not have an obvious smallness. Thus, the problem arises of closing this chain of equations. In hydrodynamics, the system of equations is closed by means of phenomenological assumptions and approximations.[29] For finite Fermi systems, which nuclei are, such approximations are, as a rule, invalid. A different approach is needed. Of course, one can solve the problem by simply omitting the coupling term, as was done, for example, in Ref. 12. However, such a method can lead to serious errors, as will be shown below. The closure problem was discussed in Ref. 31, but nothing essentially new was proposed there. An original idea for solving it was proposed in Refs. 26 and 32 and described in detail in Ref. 33. The essence of the idea is as follows. It is readily seen that the coupling terms occur in the equations in the form of a derivative with respect to x_i. Thus, if any equation is integrated over the volume, the integral of the coupling term is transformed into an integral over an infinitely distant surface, where the Wigner function f, and with it ρ, ρu_i, $P_{ij\ldots k}$ vanish. It is this fact that can be used to close the infinite system of equations.

Moments with respect to the coordinates (virial theorems)

It was shown in Refs. 13–15 that for the study of collective motion in nuclei it is very convenient to use the method of virials, taken from Ref. 16, where it was successfully used to solve similar problems in astronomy (vibrational motions of rotating self-gravitating masses). This method is attractive because one can obtain equations of motion directly for integrated properties of nuclei, in particular, for all possible multipole moments.

First virial

We integrate the first equation of the system (8)–(11) (continuity equation) over infinite volume with unit weight. The integral of $\mathrm{div}(n\mathbf{u})$ vanishes by virtue of the boundary condition, while the first term gives an obvious constant of the motion:

$$\frac{d}{dt}\int n(\mathbf{r},t)d\mathbf{r} \equiv \frac{d}{dt}A(t) = 0,$$

i.e., the number of nucleons is conserved.

Second virial

We now integrate the continuity equation over the volume with weight x_i. Integrating the second term by parts and using the boundary condition, we obtain an equation of motion for the coordinate $J_i(t) = \int x_i \rho(\mathbf{r},t)d\mathbf{r}$ of the center of mass of the nucleons:

$$\frac{d}{dt}J_i(t) - \int \rho(\mathbf{r},t)u_i(\mathbf{r},t)d\mathbf{r} = 0. \tag{12}$$

Obviously, it must be augmented by an equation of motion for the integrated quantity $\int \rho u_i d\mathbf{r}$ (the momentum of the nucleus), which is obtained by integrating Eq. (9) over the volume with unit weight:

$$\frac{d}{dt}\int \rho u_i d\mathbf{r} + \int n\frac{\partial W}{\partial x_i}d\mathbf{r} = 0. \tag{13}$$

The integral of the coupling term $\int (\partial P_{ik}/\partial x_k)d\mathbf{r}$ vanishes by virtue of the boundary condition. In the absence of external fields, the second integral in Eq. (13) is the sum of all internal forces acting in the nucleus, and it must vanish. In this case, Eq. (13) expresses the law of conservation of the momentum $\int \rho \mathbf{u} d\mathbf{r}$ of the nucleus, and Eq. (12) gives trivial information about the uniform and rectilinear motion of the nucleus as a whole.

The previous formulas admit a different treatment for the distribution function and the macroscopic variables related to it: ρ, $\mathbf{u}$, P_{ij}, etc. In particular, one can write down equations like (1) for a system of several density matrices that contain information about the nucleons of different species: protons or neutrons. In this case, the neutron–proton interaction makes a nonvanishing contribution to the integral $\int \rho(\partial W/\partial x_i)d\mathbf{r}$. Then Eqs. (12) and (13) describe the motion of the centers of gravity of the neutrons and protons relative to each other and can provide a basis for study of the giant dipole resonance.

Third virial

Integrating the continuity equation over $\mathbf{r}$ with weight $x_i x_j$, we obtain the equation of motion for the inertia tensor $J_{ij}(t) = \int x_i x_j \rho(\mathbf{r},t)d\mathbf{r}$:

$$\frac{d}{dt}J_{ij}(t) - \int (x_i u_j + x_j u_i)\rho(\mathbf{r},t)d\mathbf{r} = 0. \tag{14}$$

We see that it must be augmented by the equation of motion for the integrated variable $\int x_{ij} u(\mathbf{r},t)\rho(\mathbf{r},t)d\mathbf{r}$, which is obtained by integrating Eq. (9) over the volume with weight x_j:

$$\frac{d}{dt}\int x_j u_i \rho d\mathbf{r} + \mathscr{W}_{ij} - K_{ij} - \Pi_{ij} = 0. \tag{15}$$

Here, $\mathscr{W}_{ij} = \int x_j n(\partial W/\partial x_i)d\mathbf{r}$ is the tensor of the potential energy. In the absence of external forces, it is symmetric with respect to i and j. Further, $K_{ij} = \int \rho u_i u_j d\mathbf{r}$ is the tensor of the kinetic energy of the nucleus, and $\Pi_{ij} = \int P_{ij}(\mathbf{r},t)d\mathbf{r}$ is the integrated pressure tensor—in what follows, it will simply be called the pressure tensor. The equation of motion for Π_{ij} is obtained by integrating Eq. (10) over $\mathbf{r}$ with unit weight:

$$\frac{d\Pi_{ij}}{dt} + \sum_{k=1}^{3}\int \left(P_{ik}\frac{\partial u_{jk}}{\partial x_k} + P_{jk}\frac{\partial u_i}{\partial x_k}\right)d\mathbf{r} = 0. \tag{16}$$

The integral of the coupling term $\int (\partial P_{ijk}/\partial x_k)d\mathbf{r}$ has vanished, and we are left with the three integro-differential equations (14)–(16), which contain only the three un-

known functions n, $\mathbf{u}$, and P_{ij} [in view of the Hohenberg–Kohn theorem,[17] the self-consistent average field $W(\mathbf{r})$ can be assumed to be a functional of the density $n(\mathbf{r})$]. The tensors of higher rank P_{ijk} do not occur here, and in this sense the system of equations (14)–(16) can be regarded as closed. We recall that the indices i and j take the values 1, 2, 3, so that this system actually contains 21 equations [Eqs. (14) and (16) are symmetric in i and j]. It is convenient to recombine the nine equations of (15) in symmetric and antisymmetric blocks. Thus, adding to (15) the same equation with interchanged indices, and using (14), we find

$$\frac{d^2}{dt^2} J_{ij}(t) + 2(\mathscr{W}_{ij} - K_{ij} - \Pi_{ij}) = 0. \tag{17}$$

Then, subtracting from (15) the same equation with interchanged indices, we obtain the conservation law for the angular momentum:

$$\frac{d}{dt} \int \rho(x_j u_i - x_i u_j) d\mathbf{r} = 0. \tag{18}$$

By means of (9) and (16), we can also obtain[26] the energy conservation law:

$$\frac{d}{dt} \left\{ \frac{1}{2} \sum_{i=1}^{3} (K_{ii} + \Pi_{ii}) + \frac{1}{2} \int nW d\mathbf{r} \right\} = 0. \tag{19}$$

Fourth virial

Integration of the continuity equation with respect to $\mathbf{r}$ with weight $x_i x_j x_k$ gives an equation of motion for the integrated nuclear variable $J_{ijk}(t) = \int x_i x_j x_k \rho(\mathbf{r},t) d\mathbf{r}$:

$$\frac{d}{dt} J_{ijk}(t) - \int (x_i x_j u_k + x_i x_k u_j + x_j x_k u_i) \rho(\mathbf{r},t) d\mathbf{r} = 0. \tag{20}$$

Proceeding as in the derivation of all the preceding virial equations, we obtain the equations of motion

$$\frac{d}{dt} \int \rho u_i x_j x_k dr - K_{ij,k} - K_{ik,j} + \mathscr{W}_{i,jk} - \Pi_{ij}^{k} - \Pi_{ik}^{j} = 0; \tag{21}$$

$$\frac{d}{dt} \Pi_{ij}^{k} - \int P_{ij} u_k d\mathbf{r} + \sum_{s=1}^{3} \int \left[P_{js} \frac{\partial u_i}{\partial x_s} \right]_{ij} x_k d\mathbf{r} - \Pi_{ijk} = 0; \tag{22}$$

$$\frac{d}{dt} \Pi_{ijk} + \sum_{s=1}^{3} \int \left[P_{jks} \frac{\partial u_i}{\partial x_s} - \frac{1}{\rho} P_{jk} \frac{\partial P_{is}}{\partial x_s} \right]_{ijk} d\mathbf{r} = \frac{\hbar^2}{4m^2} \int n \frac{\partial^3 W}{\partial x_i \partial x_j \partial x_k} d\mathbf{r}. \tag{23}$$

Here $K_{ij,k} = \int \rho u_i u_j x_k d\mathbf{r}$, $\mathscr{W}_{i,jk} = \int n x_j x_k (\partial W / \partial x_i) d\mathbf{r}$, $\Pi_{ij}^{k}(t) = \int P_{ij}(\mathbf{r},t) x_k d\mathbf{r}$, $\Pi_{ijk}(t) = \int P_{ijk}(\mathbf{r},t) d\mathbf{r}$. The indices on the square brackets signify that a summation over them must be made. For example,

$$[A_{i,jk}]_{ijk} = A_{i,jk} + A_{j,ik} + A_{k,ij}.$$

Equations (20)–(23) form a closed system of integro-differential equations for ρ, $\mathbf{u}$, P_{ij}, and P_{ijk}—the higher-rank tensors $P_{ijk...l}$ do not occur in them. This system is remarkable in that for the first time a quantum correction appears in it—the term on the right-hand side of Eq. (23), which is due to the difference between the equation for the Wigner function and the Vlasov equation.

Fifth virial

The equations of motion for the fourth-rank tensors are obtained in the same way as all the preceding ones:

$$\frac{\partial}{\partial t} J_{ijkl} - \left[\int \rho u_i x_j x_k x_l d\mathbf{r} \right]_{ijkl} = 0,$$

$$\frac{\partial}{\partial t} \int \rho u_i x_j x_k x_l d\mathbf{r} - \left[\Pi_{ij}^{kl} + \int \rho u_i u_j x_k x_l d\mathbf{r} \right]_{jkl} + \int n x_j x_k x_l \frac{\partial W}{\partial x_i} d\mathbf{r} = 0,$$

$$\frac{\partial}{\partial t} \Pi_{ij}^{kl} - \left[\Pi_{ijk}^{l} + \int P_{ij} u_k x_l d\mathbf{r} \right]_{kl} + \sum_{s=1}^{3} \left[\int P_{js} \frac{\partial u_i}{\partial x_s} x_k x_l d\mathbf{r} \right]_{ij} = 0, \tag{24}$$

$$\frac{\partial}{\partial t} \Pi_{ijk}^{l} + \sum_{s=1}^{3} \left[\int P_{jks} \frac{\partial u_i}{\partial x_s} x_l d\mathbf{r} - \int \frac{1}{\rho} P_{ij} \frac{\partial P_{ks}}{\partial x_s} x_l d\mathbf{r} \right]_{ijk} - \Pi_{ijkl} - \int P_{ijk} u_l d\mathbf{r} = \frac{\hbar^2}{4m^2} \int n x_l \frac{\partial^3 W}{\partial x_i \partial x_j \partial x_k} d\mathbf{r},$$

$$\times \frac{\partial}{\partial t} \Pi_{ijkl} + \sum_{s=1}^{3} \left[\int \left(P_{jkls} \frac{\partial u_i}{\partial x_s} - \frac{1}{\rho} P_{jkl} \frac{\partial P_{is}}{\partial x_s} \right) d\mathbf{r} \right]_{ijkl} = 0.$$

We have here introduced the notation

$$\Pi_{ij}^{kl}(t) = \int P_{ij}(\mathbf{r},t) x_k x_l d\mathbf{r},$$

$$\Pi_{ijk}^{l}(t) = \int P_{ijk}(\mathbf{r},t) x_l d\mathbf{r},$$

$$\Pi_{ijkl}(t) = \int P_{ijkl}(\mathbf{r},t) d\mathbf{r},$$

$$J_{ijkl}(t) = \int \rho(\mathbf{r},t) x_i x_j x_k x_l d\mathbf{r}.$$

We do not require the systems of equations for the tensors of still higher rank. It is readily seen that the procedure for deriving them is extremely simple and can be readily continued.

Characteristic features of the virial equations of the moment method

We now discuss and evaluate our results. Initially, one partial differential equation for a function of seven variables, $f(\mathbf{r}, \mathbf{p}, t)$, was replaced by an infinite system of equations for the moments of this function in the momentum space: $n(\mathbf{r}, t)$, $u_i(\mathbf{r},t)$, $P_{i_1\ldots i_n}(\mathbf{r},t)$. This system has long been known,[24,34] but is not very popular precisely because it is infinite and there was no satisfactory prescription for working with it (well-founded prescription for truncating it).

Then, in place of the infinite system of equations for the functions n, u_i, $P_{i_1\ldots i_k}$, we have introduced another infinite system of equations, but now for the moments of these functions in the coordinate space, i.e., for all possible moments of the functions f in the phase space $(\mathbf{r}, \mathbf{p})$. As regards the infinity, this system is, of course, no better than its predecessor. But it does possess a remarkable feature which makes it exceptionally convenient for practical applications—it decomposes into independent finite subsystems of integro-differential equations for the functions $n(\mathbf{r}, t)$, $\mathbf{u}(\mathbf{r}, t)$, $P_{i_1i_2}(\mathbf{r},t),\ldots,P_{i_1\ldots i_k}$. However, a question arises: If we find solutions of the subsystems for n, $\mathbf{u}$, . . ., $P_{i_1\ldots i_k}$ and for n, $\mathbf{u}$, . . ., $P_{i_1\ldots i_{k+1}}$, will the functions n, $\mathbf{u}$, etc., found by solving the different subsystems be the same? One can say with confidence that most probably they will not. Is there a contradiction here? No! For the solution of each individually taken subsystem is not unique. Only a solution that satisfies all (i.e., an infinite number) of subsystems can be unique. Obviously, it is equivalent to the solution of the original differential equation.

From the formal point of view, all these subsystems are simply sets of dynamical integral relations that the exact Wigner function must satisfy. We are interested in them only to the extent that they describe the moments of this function. Indeed, knowledge of all moments of the function f is equivalent to knowledge of the function itself. If our problem is such that its solution requires knowledge of only some moments of the function, then it is quite unnecessary to find this function itself in order to calculate the required moments. It is natural to attempt to write down equations of motion directly for the moments. It is obvious that our subsystems are the required equations of motion. However, when these subsystems are viewed in this way it turns out that they cannot be regarded as completely independent, and one can speak merely of approximate independence. Let us consider in more detail a subsystem, for example, (14)–(16). What does it describe? If we judge from the terms with time derivatives, then, of course, it describes the dynamics of the tensors J_{ij}, $\int x_i u_j \rho d\mathbf{r}$, and Π_{ij}, which can be called collective variables of the problem. Such an answer will be completely rigorous only if all the tensors present in the equations, $\mathscr{W}_{ij}$, K_{ij}, $\int P_{ik}(\partial u_j/\partial x_k)d\mathbf{r}$, can be expressed in terms of the three variables just listed. This can indeed be done for the harmonic oscillator. To see this, it is sufficient to rewrite Eq. (11) in the equivalent form

$$\frac{\partial}{\partial t}\widetilde{P}_{i_1\ldots i_n}+\sum_{s=1}^{3}\frac{\partial}{\partial x_s}\widetilde{P}_{i_1\ldots i_n s}+\frac{1}{m}\sum_{t=1}^{n}\frac{\partial W}{\partial x_{i_t}}\widetilde{P}_{i_1\ldots(i_t)\ldots i_n}+(\text{q.c.})=0, \tag{25}$$

where $\widetilde{P}_{i_1\ldots i_n}(\mathbf{r},t)=\int p_{i_1}\ldots p_{i_n}f(\mathbf{r},\mathbf{p},t)d\mathbf{p}$, and the expression for the quantum correction (q.c.) is the same as in (11) (with replacement of $P_{ij\ldots k}$ by $\widetilde{P}_{ij\ldots k}$). If $W=\alpha r^2$, then the quantum correction disappears, and the equivalent of the subsystem (14)–(16) takes the closed form

$$\frac{d}{dt}J_{ij}-\widetilde{\Pi}_i^j-\widetilde{\Pi}_j^i=0,$$

$$\frac{d}{dt}\widetilde{\Pi}_i^j+\frac{2\alpha}{m}J_{ij}-\widetilde{\Pi}_{ij}=0,$$

$$\frac{d}{dt}\widetilde{\Pi}_{ij}+\frac{2\alpha}{m}(\widetilde{\Pi}_i^j+\widetilde{\Pi}_j^i)=0.$$

It is interesting to note that in the case of a Coulomb or gravitational interaction between the particles the average field within an object with a homogeneous particle distribution (i.e., $n=\text{const}$) has precisely a quadratic dependence on the radius. It is for precisely this reason that the moment method proved so effective in astronomy.[16] In the case of an arbitrary interaction, the problem posed above does not have an exact solution. If, say, we represent the average field $U(\mathbf{r})$ in the form of a polynomial in $\mathbf{r}$, then all terms $\mathbf{r}k$ with $k>2$ will introduce into the subsystem (14)–(16) tensors of rank higher than two, and thus the given subsystem is coupled to all the others. But this is by no means the decoupling that exists in the case of the system of equations for the functions n, u_i, $P_{ij\ldots k}$. For here it is clear that any truncation of the system of equations for the moments is equivalent to some approximate description of the difference between an oscillator field and the true average field, and, moreover, the correctness of the approximation can be checked. Thus, although the existence of the almost independent subsystems does not amount to any gain from the purely mathematical point of view, it is convenient from the practical point of view. The closure problem, in which it was not even clear how one should attack it in the case of the infinite hierarchy of equations for the functions n, u_i, $P_{ij\ldots k}$, is now reduced to the more or less familiar problem of finding an acceptable approximation for the potential. Its solution, in turn, is determined by the physics of the investigated phenomenon.

The problem of transforming almost independent subsystems into truly independent ones can be solved comparatively easily if one is studying small deviations of a nucleus from the equilibrium state (small-amplitude motions, when the equations can be linearized with respect to the amplitudes), to which the main attention in this review will be devoted.

It should also be emphasized that closed systems of equations can be obtained for the tensors $\widetilde{\Pi}^{i_{k+1}\ldots i_n}_{i_1\ldots i_k}=\int\int f(\mathbf{r},\mathbf{p},t)p_{i_1}\ldots p_{i_k}x_{i_{k+1}}\ldots x_{i_n}d\mathbf{p}d\mathbf{r}$, where k takes all values from 0 to n, with, moreover, $\widetilde{\Pi}^{i_1\ldots i_n}\equiv J_{i_1\ldots i_n}/m$, and $\widetilde{\Pi}^{i_2\ldots i_n}_{i_1}=\int\rho u_{i_1}x_{i_2}\ldots x_{i_n}d\mathbf{r}$, $\widetilde{\Pi}^{i_3\ldots i_n}_{i_1i_2}=\int\widetilde{P}_{i_1i_2}x_{i_3}\ldots x_{i_n}d\mathbf{r}$, etc. There-

fore, the dynamics of the inertia tensor of rank n, $J_{i_1\ldots i_n}$, can be described only in conjunction with all the tensors $\widetilde{\Pi}^{i_{k+1}\ldots i_n}_{i_1\ldots i_k}$ $(1\leqslant k\leqslant n)$. The tensors $P_{i_1\ldots i_k}$ $(k\geqslant 2)$ describe the deformation of the Fermi surface of multipolarity $2\leqslant\lambda\leqslant k$. Therefore one can say that to describe the evolution of the moment of the nucleus (in the coordinate space) of multipolarity λ it is necessary to take into account the deformations of the Fermi surface of all multipolarities up to λ.

We note finally that in such an approach the kinetic-energy operator is treated exactly, all the approximations being related exclusively to the average field. This is important, because the kinetic-energy operator is much more sensitive to all approximations, and they can, accordingly, lead to large errors.

Small deviations from equilibrium

The virial equations can be readily adapted to describe small-amplitude motions. For this, it is necessary to vary them in accordance with the rules of variation of integrated quantities.[16] In the case of an infinite volume of integration, these rules are extremely simple—it is necessary to replace n, u_i, $P_{ij\ldots k}$ in the integrands by their Eulerian variations δn, δu_i, $\delta P_{ij\ldots k}$. Thus, the variation of the fourth virial, after neglect of the terms quadratic in the variations, takes the form

$$\frac{d}{dt}\delta J_{ijk}-\left[\int x_ix_j(u_k^{(0)}\delta\rho+\rho^{(0)}\delta u_k)d\mathbf{r}\right]_{ijk}=0;$$

$$\frac{d}{dt}\int(\rho^{(0)}\delta u_i+u_i^{(0)}\delta\rho)x_jx_kd\mathbf{r}-\delta K_{ij,k}-\delta K_{ik,j}+\delta\mathscr{W}_{i,jk}-\delta\Pi_{ij}^k-\delta\Pi_{ik}^j=0;$$

$$\frac{d}{dt}\delta\Pi_{ij}^k-\int(P_{ij}^{(0)}\delta u_k+u_k^{(0)}\delta P_{ij})d\mathbf{r}+\sum_{s=1}^{3}\int\left[P_{js}^{(0)}\frac{\partial\delta u_i}{\partial x_s}+\frac{\partial u_i^{(0)}}{\partial x_s}\delta P_{js}\right]_{ij}x_kd\mathbf{r}-\delta\Pi_{ijk}=0;$$

$$\frac{d}{dt}\delta\Pi_{ijk}+\sum_{s=1}^{3}\int\left[P_{jks}^{(0)}\frac{\partial\delta u_i}{\partial x_s}+\frac{\partial u_i^{(0)}}{\partial x_s}\delta P_{jks}+\frac{\delta\rho}{(\rho^{(0)})^2}P_{jk}^{(0)}\frac{\partial P_{is}^{(0)}}{\partial x_s}-\frac{1}{\rho^{(0)}}\delta P_{jk}\frac{\partial P_{is}^{(0)}}{\partial x_s}-\frac{1}{\rho^{(0)}}P_{jk}^{(0)}\frac{\partial\delta P_{is}}{\partial x_s}\right]_{ijk}d\mathbf{r}=\delta\chi_{ijk}. \tag{26}$$

Here $\delta\chi_{ijk}=(\hbar^2/4m^2)\int(\delta n\partial^3W^{(0)}/\partial x_i\partial x_j\partial x_k+n^{(0)}\partial^3\delta W/\partial x_i\partial x_j\partial x_k)d\mathbf{r}$, $\delta K_{ij,k}=\int(\rho^{(0)}u_j^{(0)}\delta u_i+\rho^{(0)}u_i^{(0)}\delta u_j+u_i^{(0)}u_j^{(0)}\delta\rho)x_kd\mathbf{r}$, $\delta\mathscr{W}_{i,jk}=\int x_jx_k(\delta n\partial W^{(0)}/\partial x_i+n^{(0)}\partial\delta W/\partial x_i)d\mathbf{r}$. The superscript (0) means that the given quantity is taken in the equilibrium state. In what follows, we shall consider only equilibrium states with $u_i^{(0)}=P_{ijk}^{(0)}=0$, so that Eqs. (26) are somewhat simplified.

The variations δn and δu_i are not independent. A relation between them can be obtained by means of the continuity equation:[16]

$$\delta n=-\operatorname{div}(n\boldsymbol{\xi}),\quad \delta u_i=\frac{\partial\xi_i}{\partial t}, \tag{27}$$

where $\xi_i(\mathbf{r},t)\equiv dx_i$ is an infinitesimal displacement.

We also introduce the collective variables

$$V_{i,j\ldots k}=\int\rho\xi_ix_j\ldots x_kd\mathbf{r}. \tag{28}$$

They are natural for our problem, since the variation of the inertia tensor is a linear combination of them:

$$\delta J_{ij\ldots k}=\int x_ix_j\ldots x_k\delta\rho d\mathbf{r}=\sum_{s=1}^{3}\int\rho\xi_s\frac{\partial}{\partial x_s}(x_ix_j\ldots x_k)d\mathbf{r}=V_{ij\ldots k}, \tag{29}$$

where $V_{ij\ldots k}=[V_{i,j\ldots k}]_{ij\ldots k}$. Substituting (27) in (26), we see that the first equation is satisfied identically, while the remaining ones take the form

$$\frac{d^2}{dt^2}V_{i,jk}+\delta\mathscr{W}_{i,jk}-\delta\Pi_{ij}^k-\delta\Pi_{ik}^j=0,$$

$$\frac{d}{dt}\delta\Pi_{ij}^k-\frac{d}{dt}\int P_{ij}^{(0)}\xi_kd\mathbf{r}+\sum_{s=1}^{3}\frac{d}{dt}\int\left[P_{js}^{(0)}\frac{\partial\xi_i}{\partial x_s}\right]_{ij}x_kd\mathbf{r}-\delta\Pi_{ijk}=0,$$

$$\frac{d}{dt}\delta\Pi_{ijk}-\sum_{s=1}^{3}\int\frac{1}{\rho^{(0)}}\left[\frac{1}{\rho^{(0)}}P_{jk}^{(0)}\frac{\partial P_{is}^{(0)}}{\partial x_s}\operatorname{div}(\rho\boldsymbol{\xi})+\delta P_{jk}\frac{\partial P_{is}^{(0)}}{\partial x_s}+P_{jk}^{(0)}\frac{\partial\delta P_{is}}{\partial x_s}\right]_{ijk}d\mathbf{r}=\delta\chi_{ijk}. \tag{30}$$

The variations $\delta\mathscr{W}_{i,jk}$ and $\delta\chi_{ijk}$ depend on the specific form of the self-consistent average field and are, obviously, functionals of $\boldsymbol{\xi}$. The collective variables in (30) are $V_{i,jk}$, $\delta\Pi_{ij}^k$, and $\delta\Pi_{ijk}$. The problem now is to express all the integrals in (30) in terms of these variables.

As we have already said, in the general case this problem does not have an exact solution. However, there exist two factors that enable us to construct an approximate solution that, in principle, will be arbitrarily close to the exact solution. These are the finite size of the nucleus and the small amplitude of the studied motions. The second factor was already exploited in the derivation of the linearized system of equations (26). The importance of the first factor becomes obvious if we represent the unknown function $\xi_i(\mathbf{r},t)$ in the form of the series

$$\xi_i(\mathbf{r},t)=L_i(t)+\sum_{j=1}^{3}L_{i,j}(t)x_j+\sum_{j,k=1}^{3}L_{i,jk}(t)x_jx_k+\sum_{j,k,l=1}^{3}L_{i,jkl}(t)x_jx_kx_l+\ldots, \tag{31}$$

where $L_{i,jk...l}$ are unknown functions of the time. How many terms of this series are important for construction of the solution? The answer obviously depends on the type of motion that is studied. Practically all the existing models (Refs. 5, 31, and 35–40) predict a radial dependence of ξ in the form of a Bessel function, which corresponds formally to an infinite series (31). However, on closer examination it turns out that only a very small part of this function, with a small number of characteristic points (maxima, minima, zeros) or none at all is situated within the nucleus. It is clear that this "piece" of the Bessel function can be approximated to an arbitrarily good accuracy by a finite number of terms of the series (31). Bessel functions were mentioned only as an example—the main criterion for determining the necessary number of terms of the series must of course be agreement with the experimental data and internal consistency of the proposed scheme of calculation.

We clarify this last remark. Suppose we have decided to restrict ourselves to the first three terms of the series (31). We integrate it over $\mathbf{r}$ with weights ρ, ρx_j, $\rho x_j x_k$. As a result, we obtain a system of linear equations that enables us to express the variables V_i, $V_{i,j}$, and $V_{i,jk}$ uniquely in terms of L_i, $L_{i,j}$, and $L_{i,jk}$ and vice versa. It follows from this that the approximation adopted for ξ_i obliges us to write down dynamical equations for all the tensors V_i, $V_{i,j}$, $V_{i,jk}$, since otherwise the system would be incomplete. It is easy to formulate a general rule: If in the series (31) the terms up to $L_{i,j_1j_2...j_n}$ are retained, then it is necessary to have equations for all tensors from V_i to $V_{i,j_1j_2...j_n}$. The converse is also true: If we have written down a system of dynamical equations for the tensors from V_i to $V_{i,j_1j_2...j_n}$, then in the series (31) it is necessary to retain all terms from L_i to $L_{i,j_1j_2...j_n}$. If we keep a smaller number of tensors $L_{i,jk...l}$, then from the system of equations that relate $V_{i,j...k}$ and $L_{i,j...k}$ we immediately find that not all tensors $V_{i,j...k}$ are independent and, therefore, the system of dynamical equations for $V_{i,j...k}$ is overdetermined. Thus, we have shown that there exists a one-to-one correspondence between the number of terms of the series (31) taken into account and the number of moments of the Wigner function for which dynamical equations must be written down. The more complicated (in the sense of the picture of the flow displacement distribution) motion that we wish to investigate, the larger the number of moments that must be taken into account. Naturally, the converse is also true: The greater the number of moments covered by the equations of motion, the more complicated the motion of the nuclear matter that we can describe.

All these arguments can be repeated practically unchanged for the expansion of $\delta P_{ij}(\mathbf{r},t)$ as well:

$$\delta P_{ij}(\mathbf{r},t)=n^{(0)}(\mathbf{r})\left[D_{ij}(t)+\sum_{k=1}^{3}D_{ij,k}(t)x_k+\sum_{k,l=1}^{3}D_{ij,kl}(t)x_kx_l+\ldots\right]. \tag{32}$$

The factor $n^{(0)}(\mathbf{r})$ is introduced here with the obvious aim of guaranteeing vanishing of δP_{ij} outside the nucleus. In the case of the expansion of ξ, this was not needed, since ξ never appears in an integrand without $n^{(0)}$ or $P^0_{ij...k}$. Integrating (32) over $\mathbf{r}$ with weights 1, x_k, x_kx_l..., we can obtain a system of equations that uniquely relate the functions $D_{ij,k...l}(t)$ to the tensors $\delta\Pi_{ij}^{k...l}$.

We now note an important detail. Expanding the tensors $\delta\Pi_{ij...k}^{l...n}(\mathbf{r},t)$ in series in x_i and truncating them, we thus make certain approximations for the Wigner function in the coordinate space—in the momentum space it is treated exactly, without any approximations. This is the fundamental difference (and advantage) of our method from the well-known "scaling approximation" (Refs. 5, 9–11, 35, 36, 39, and 41–46).

It now merely remains to substitute the first three terms of the series (31) and the first two terms of (32) in Eqs. (30), add to them the variations of the systems of equations (14)–(16) and (12), (13), and show that we have obtained a closed system of equations for the tensors V_i, $V_{i,j}$, $V_{i,jk}$, $\delta\Pi_{ij}$, $\delta\Pi_{ij}^k$, and $\delta\Pi_{ijk}$. In what follows, we shall consider only nuclei with triplanar symmetry (in the equilibrium state), so that the equations for the tensors of even and odd ranks decouple.

2. VIRIAL EQUATIONS FOR SECOND-RANK TENSORS; APPROXIMATION OF AN INCOMPRESSIBLE NUCLEUS WITH SHARP EDGE

In this section, we shall study quadrupole vibrations of rotating nuclei by means of the virial equations (14)–(16).

To this end, we write them in a rotating coordinate system:

$$\frac{d}{dt}J_{ij}-\int\rho(x_iu_j+x_ju_i)d\mathbf{r}=0,$$

$$\frac{d}{dt}\int\rho x_ju_i d\mathbf{r}+\mathscr{W}_{ij}-K_{ij}+\Pi_{ij}+2\sum_{s,k=1}^{3}\varepsilon_{isk}\Omega_s\times\int\rho u_kx_jd\mathbf{r}+\sum_{k=1}^{3}\Omega_i\Omega_kJ_{jk}-\Omega^2J_{ij}=0, \tag{33}$$

$$\frac{d}{dt}\Pi_{ij}+\sum_{k=1}^{3}\int\left(P_{ik}\frac{\partial u_j}{\partial x_k}+P_{jk}\frac{\partial u_i}{\partial x_k}\right)d\mathbf{r}+2\sum_{s,k=1}^{3}\Omega_s[\varepsilon_{isk}\Pi_{kj}]_{ij}=0,$$

where ε_{ijk} is the Levi-Civita symbol, and Ω_i are the components of the angular velocity.

These equations enable us to investigate both the equilibrium state of the rotating nucleus (the state of "secular equilibrium"), and small deviations from it—the giant quadrupole resonance and the low-lying modes. To get a feel for the way in which the method works, and to demonstrate its possibilities, we shall work in the approximation of a sharp edge of a nucleus with homogeneous density, treating the nuclear matter as incompressible. This will enable us to obtain practically all the results analytically; moreover. such an approximation is entirely adequate for many physical problems.

Secular equilibrium and shape of rotating nuclei

By equilibrium we here understand a state described by Eqs. (33) with no time dependence. In astronomy, such a state of rotating bodies is called secular equilibrium. For $\Omega = 0$, this would be the ground state of the nucleus. We shall consider the simplest case when the nucleus rotates as a whole, i.e., in the equilibrium state $u_i^{(0)} = 0$. Then the kinetic-energy tensor K_{ij} vanishes. The potential-energy tensor consists of nuclear and Coulomb parts:

$$\mathscr{W}_{ij} = \mathscr{U}_{ij} + C_{ij},$$

where $\mathscr{U}_{ij} = \int n x_j (\partial U/\partial x_i) d\mathbf{r}$, U is the average nuclear field, and

$$C_{ij} = \left(\frac{Ze_p}{A}\right)^2 \int d\mathbf{r} n(\mathbf{r}) x_j \frac{\partial}{\partial x_i} \int d\mathbf{r}' \frac{n(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$= -\left(\frac{Ze_p}{A}\right)^2 \frac{1}{2} \iint n(\mathbf{r}) n(\mathbf{r}') \times \frac{(x_i - x_i')(x_j - x_j')}{|\mathbf{r}-\mathbf{r}'|^3} d\mathbf{r} d\mathbf{r}'.$$

The tensor $\mathscr{U}_{ij}$ can decomposed into surface and volume parts:

$$\mathscr{U}_{ij} = n_0 \int_V x_j \frac{\partial U}{\partial x_i} d\mathbf{r}$$

$$= n_0 \left\{ \int_S x_j U s_i d\mathbf{S} - \delta_{ij} \int_V U d\mathbf{r} \right\}$$

$$\equiv 2\sigma_{ij} - \delta_{ij} \mathscr{U}.$$

Here, $\int_V$ denotes integration over the volume in which the density n is nonzero, and s_i are the components of the unit vector of the outer normal to the surface S, which bounds the given volume. We direct the rotation axis along the x_3 axis, so that $\Omega_3 = \Omega$, $\Omega_2 = \Omega_1 = 0$. The equilibrium equations (33) take the form

$$2\sigma_{ij} - \delta_{ij}\mathscr{U}_0 + C_{ij} - \Pi_{ij} + \Omega^2(\delta_{i3} - 1)J_{ij} = 0, \tag{34}$$

$$\Omega \sum_{s=1}^{3} (\varepsilon_{i3s}\Pi_{sj} + \varepsilon_{j3s}\Pi_{si}) = 0. \tag{35}$$

Equation (34) represents the balance of the pressure, Coulomb, nuclear (surface and volume), and centrifugal forces.

We shall mainly consider nuclei that have spherical shape in the ground state (for $\Omega = 0$). Deformation occurs in them only because of rotation. The effectiveness of the method is largely determined by the choice of the parameters that fix the shape of the nucleus: The greater the number of parameters required to reproduce accurately the precise shape of the nuclear surface, the larger the number of equations (moments) that must be used for the calculations. From the mathematical point of view, the most fully developed approximation of the shape of the surface is by ellipsoids,[16,47–50] which we shall use. We relate the semiaxes a_i of the ellipsoid to the deformation parameter δ and nonaxiality parameter γ as follows:

$$a_{1,2}^2 = a_0^2\left(1 - \frac{4}{3}\delta\cos(\gamma \pm 120°)\right),$$

$$a_3^2 = a_0^2\left(1 - \frac{4}{3}\delta\cos\gamma\right). \tag{36}$$

The parameter a_0 can be expressed in terms of δ and γ by means of the condition of conservation of the volume: $a_1a_2a_3 = R^3 = r_0^3 A$, where $r_0 = 1.2$ F, and R is the radius of the sphere.

For such a parametrization of its shape, the nucleus possesses triplanar symmetry, and the integrals σ_{ij}, C_{ij}, and J_{ij} are nonzero only for $i = j$. But it then follows from (34) that the tensor Π_{ij} must be diagonal: $\Pi_{ij} = \delta_{ij}\Pi_{ii}$.

Thus, the nondiagonal components of the tensor (34) can be satisfied identically, and only the diagonal components remain. Subtracting the component with $i = 3$ from the components with $i = 1$ and $i = 2$, we obtain two relations between the shape parameters δ and γ and the angular velocity Ω:

$$\Omega^2 J_{11} = 2(\sigma_{11} - \sigma_{33}) + (C_{11} - C_{33}) + (\Pi_{33} - \Pi_{11}),$$

$$\Omega^2 J_{22} = 2(\sigma_{22} - \sigma_{33}) + (C_{22} - C_{33}) + (\Pi_{33} - \Pi_{22}). \tag{37}$$

Note that the volume part of the nuclear potential does not occur here at all. Further, it follows from Eq. (35) that $\Pi_{11} = \Pi_{22}$. There are also no restrictions on Π_{33}. Therefore, we can set $\Pi_{11} = \Pi_{22} = \Pi_{33} = \Pi_0$. At the least, this does not contradict common sense, which suggests that in nuclei without static deformation the pressure must be isotropic. Then the pressure tensor also disappears from the relations (37), and we arrive at the conclusion that the shape of the rotating nucleus is determined by the competition between the surface, Coulomb, and centrifugal forces, in complete agreement with the liquid-drop model of the nucleus.[49]

Once we have adopted the approximation of a sharp edge of the nucleus, then, to be consistent, it is sensible to parametrize the nuclear potential on the surface of the nucleus by a surface tension:[49]

$$nU|_S = T \operatorname{div} \mathbf{s} + \text{const}. \tag{38}$$

The coefficient of surface tension is $T = b/4\pi r_0^2$, where $b \simeq 17$ MeV is the coefficient of the surface ($\sim A^{2/3}$) term in the von Weizsäcker formula. Naturally, the constant term drops out of the difference formulas (37). With such a parametrization, the tensor of the surface forces takes the form[49]

$$\sigma_{ij} = \frac{1}{2} T \int_S (\delta_{ij} - s_i s_j) d\mathbf{S}. \tag{39}$$

Expressions for σ_{ij} and C_{ij} in terms of the ellipsoid semiaxes were derived in Refs. 16 and 49. They are given in Appendix 1. Using them, and the fact that $J_{ii} = (1/5)mAa_i^2$, we obtain the final expressions

$$\tilde{\Omega}^2 a_i^2 = \mathscr{A}_3 - \mathscr{A}_i + 2X(a_3^2 A_3 - a_i^2 A_i), \quad i = 1,2. \tag{40}$$

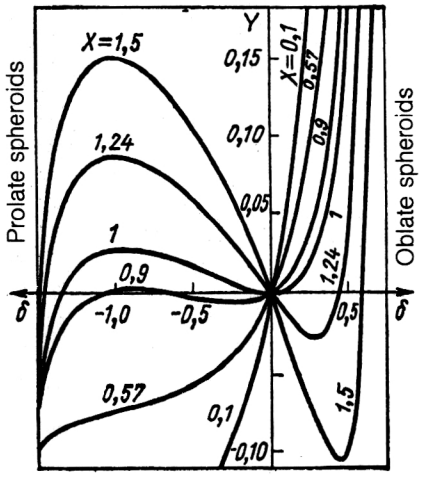

FIG. 1. Dependence of the theoretically known rotational fission parameter $Y = I^2/2I_0^2$ of a nucleus of spheroidal shape on the deformation parameter δ for different values of the fissility parameter X: $I = J\Omega$, $I_0 = \frac{2}{5}mAR^2\Omega_0$, $\Omega_0^2 = 10\pi T/mA$, $J = \frac{1}{5}mA(a_1^2 + a_2^2)$.

Here, $\tilde{\Omega}_3^2 = \Omega^2/\Omega^2$, $\Omega_3^2 = 10\pi T/mA$, $X = 2\pi q^2R^3/15T \simeq 0.0206Z^2/A$ is the fissility parameter, $q = e_pnZ/A$ is the charge density, e_p is the proton charge, and $\mathscr{A}_i$ and A_i are single-index symbols which can be expressed in terms of elliptic integrals.

The multiple-index symbols $\mathscr{A}_{ij}...$, $\mathscr{B}_{ij}...$, $A_{ij}...$, $B_{ij}...$ were defined in Refs. 16 and 49 (see Appendix 1).

As an example of the solution of Eqs. (40), Fig. 1 shows the rotation parameter (ratio of the rotational energy to the surface energy) Y as a function of the deformation for oblate and prolate spheroids and different X. Equations (40) describe numerous physical phenomena, including bifurcation, giant backbending, forebending, shape isomers,... . A complete analysis of their solutions can be found in Refs. 14, 15, and 26.

Small deviations from the equilibrium state; 2^+ excitations of rotating nuclei

It was shown in Sec. 1 that to describe various small-amplitude motions one can use the variations of the corresponding virial equations. For quadrupole vibrations, these were Eqs. (33). Varying them in accordance with the rules given there, and retaining only the terms linear in the variations, we find

$$\frac{d^2}{dt^2}V_{i,j} - \pi_{ij} - \Omega^2 V_{ij}(1-\delta_{i3}) + 2\Omega\sum_{k=1}^{3}\varepsilon_{i3k}\dot{V}_{k,j} + \delta\mathscr{W}_{ij} = 0,$$

$$\frac{d}{dt}\pi_{ij} + \sum_{k=1}^{3}\int\left(P_{ik}\frac{\partial\dot{\xi}_j}{\partial x_k} + P_{jk}\frac{\partial\dot{\xi}_i}{\partial x_k}\right)d\mathbf{r} + 2\Omega\sum_{k=1}^{3}(\varepsilon_{i3k}\pi_{kj} + \varepsilon_{j3k}\pi_{ki}) = 0. \quad (41)$$

Here $\pi_{ij} = \delta\Pi_{ij}$.

Further, following the same rules, we retain in the expansion (31) only the first two terms. The first does not contribute to any of the integrals (41) because of the triplanar symmetry of the nucleus, so that we can take

$$\xi_i = \sum_{s=1}^{3} L_{i,s}x_s. \quad (42)$$

Integrating this equation over the volume with weight ρx_j, we find a simple relation between $V_{i,j}$ and $L_{i,j}$:

$$V_{i,j} = L_{i,k}J_{kj} = L_{i,k}mA\langle x_k x_j\rangle = L_{i,j}\frac{1}{5}mAa_i^2.$$

With allowance for (42) Eqs. (41) become a system of differential equations:

$$\ddot{V}_{i,j} - \pi_{ij} - \Omega^2(1-\delta_{i3})V_{ij} + 2\Omega\sum_{k=1}^{3}\varepsilon_{i3k}\dot{V}_{k,j} + 2\delta\sigma_{ij} + \delta C_{ij} - \delta_{ij}\delta\mathscr{U}_0 = 0,$$

$$\dot{\pi}_{ij} + \frac{5\Pi_0}{mA}\left(\frac{\dot{V}_{i,j}}{a_j^2} + \frac{\dot{V}_{j,i}}{a_i^2}\right) + 2\Omega\sum_{k=1}^{3}(\varepsilon_{i3k}\pi_{kj} + \varepsilon_{j3k}\pi_{ki}) = 0. \quad (43)$$

Expressions for δC_{ij} and $\delta\sigma_{ij}$ in the approximation of a sharp edge of the nucleus were obtained in Refs. 16 and 49 and are given in Appendix 1.

The collective coordinates π_{ij} are the variation of the pressure tensor P_{ij} due to the change in the velocity distribution of the nucleons. If it is assumed that in the equilibrium state the nucleons fill a certain sphere in the velocity space uniformly (Fermi-gas model), then for the equilibrium value of the pressure we obtain

$$\Pi_0 = \frac{1}{5}mAv_F^2,$$

where $v_F = \hbar(9\pi)^{1/3}/2mr_0$ is the Fermi velocity. Small deviations from the equilibrium state for the velocity space mean small displacement dv, which can lead to only a small change in the shape of the Fermi surface. It is natural that this entire procedure has become known as the "distorted-Fermi surface model."[11,12] Here, it is necessary to note a fundamental difference between this model and the theory of a classical liquid drop. The Euler equation for the latter also contains the pressure. But there it is not an independent variable but can be expressed by means of the equation of state (a static equation!) in terms of other variables that describe the classical liquid drop (for example, in terms of the density in barotropic processes). But here the pressure tensor is an independent variable and is determined by a dynamical equation that must be solved simultaneously with the Euler equation. Thus, additional degrees of freedom appear, and as a result the spectrum of the droplet of Fermi liquid, which the nucleus is, is richer than the spectrum of the classical liquid drop.

It is helpful to recombine the 15 equations (43) in such a way as to form a system of equations in terms of irreducible tensors. It is well known[51] that from the nine components of an arbitrary Cartesian second-rank tensor (such as $V_{i,j}$ is) one can construct one tensor of zeroth rank, one pseudotensor of first rank, and one irreducible tensor of second rank. This means that Eqs. (43) must describe

0^+, 1^+, and 2^+ modes. However, they actually give information only about the 2^+ modes, since the approximation of incompressibility adopted here precludes the 0^+ mode from consideration, while 1^+ excitations do not arise because of the conservation of the angular momentum of the nucleus. Relations between the tensors $V_{i,j}$ and the irreducible tensors are given in Appendix 2.

The system of equations (43) decomposes into two independent subsystems, which describe excitations of positive and negative signature. For $a_1 = a_2$, the first subsystem, in its turn, decomposes into two parts, which describe β and γ vibrations.

Information about the normal modes can be obtained by investigating periodic solutions of the subsystems with time dependence $e^{i\omega t}$. The normal frequencies are found from the corresponding characteristic equations. As an example, we give the characteristic equation for the γ vibrations:

$$\mathscr{F}_\gamma(\omega,\Omega)\mathscr{F}_\gamma(\omega,-\Omega)=0,$$

where

$$\mathscr{F}_\gamma(\omega,\Omega)=\omega^3-6\Omega\omega^2+2\omega(5\Omega^2-d_3-g)+8\Omega(d_3-\Omega^2),$$

$$d_3^2=\Omega_0^2(\mathscr{A}_{11}-2XB_{11}),\quad g=v_F^2/a_1^2.$$

When $\Omega=0$, we obtain a single fivefold degenerate level:

$$\hbar\omega_{\text{sph}}=\hbar\sqrt{2}\left[\frac{v_F^2}{R^2}+8\Omega_0^2\frac{1-X}{15}\right]^{1/2}. \tag{44}$$

It describes well the experimental positions of the centroids of the giant quadrupole resonances in nuclei. The contribution of the deformation of the Fermi surface to ω_{sph} corresponds to the first term in the brackets; in order of magnitude it is larger than the second term, indicating a weak influence of the Coulomb and surface forces on the giant quadrupole resonance. If these forces are ignored, the results of Refs. 16 and 12 are reproduced:

$$\hbar\omega_{\text{sph}}=64.7A^{-1/3}\ \text{MeV}.$$

Figure 2 gives solutions of Eqs. (43) for $\Omega\neq0$ for the nucleus ^{154}Er ($X=0.62$).

The giant quadrupole resonance breaks up into five branches due to the appearance of deformation of the nucleus and the action of Coriolis forces. Two low-lying modes appear. In the spheroids of each branch of the giant quadrupole resonance one can determine a definite value of the projection of the multipole moment of the excitation onto the rotation axis: $\mu=0,\ \pm1,\ \pm2$.

The normal modes of the vibrations about the ellipsoidal equilibrium shape are fitted to the corresponding modes of the spheroids at the bifurcation point, where $\Omega_b^2=d_3$.

It can be seen that the excitation energy of the low-lying γ mode can take negative values. This is due to the fact that for $I > I_\delta$ the equilibrium ellipsoidal shape becomes energetically more advantageous than the spheroidal shape. The presence of the low-lying γ mode indicates that the Fermi liquid drop is soft with respect to nonaxial deformation.

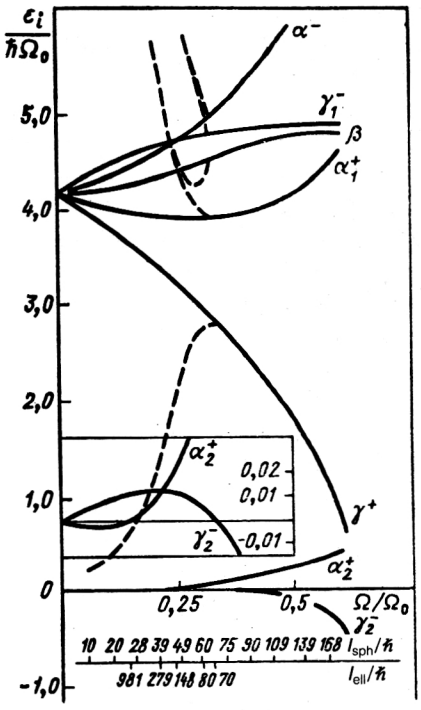

FIG. 2. Dependence of the spectrum of quadrupole vibrations on the angular velocity for ^{154}Er ($X=0.62$). The continuous curves correspond to oblate spheroids, and the broken curves to ellipsoids; $\hbar\Omega_0=2.83$ MeV. The indices + and − indicate the sign of the projection of the multipole moment. The additional abscissa gives the angular momenta corresponding to the given angular velocity of the spheroid (I_{sph}) or ellipsoid (I_{ell}).

The excitation energy of the second low-lying branch (α mode) vanishes at $\Omega=\Omega_\alpha=(1-a_3^2/a_1^2)2v_F/a_3$, to which there corresponds $I_\alpha\simeq0.4A(1-X)$. When I is somewhat greater than I_α, it can be approximated by the formula

$$\hbar\omega_\alpha\simeq\hbar\Omega(1-M_3/M_1),$$

which is characteristic of the energy of the precession mode of the rigid body[52] with the same mass distribution (M_i are the moments of inertia with respect to the axes i). Thus, allowance for the Fermi motion endows the drop with properties characteristic of a rigid body.

A deeper analysis of the physical properties of the quadrupole normal modes is impossible without knowledge of the probabilities of their excitation. They can be calculated in three ways. The first is to quantize the equations of motion (43), as was done in Refs. 25 and 26. The second is to use the classical expressions for the transition intensities.[53] The third method is the most suitable for our approach; it is to apply the theory of the linear response of the system to perturbation of it by an external field

$$O(t)=Oe^{-i\omega t}+O^+e^{i\omega t}.$$

A convenient form of response theory was given by Lane.[54] The matrix elements of the operator O satisfy the relation

$$|\langle\psi_a|O|\psi_0\rangle|^2=\lim_{\omega\to\omega_a}\hbar(\omega-\omega_a)\langle\overline{\psi_0'|Oe^{-i\omega t}|\psi_0'}\rangle, \tag{45}$$

where ψ_0 and ψ_a are the unperturbed wave functions of the stationary ground and excited states; ψ_0' is the perturbed wave function of the ground state; $\omega_a=(E_a-E_0)/\hbar$ are the normal frequencies of the system; the bar denotes averaging over a time interval much greater than $1/\omega$; ω is the frequency of the external field $O(t)$.

To use Eq. (45), it is necessary to solve two problems:

1) to express the matrix element $\langle\psi_0'|O|\psi_0'\rangle$ in terms of integral variables of the system, i.e., $V_{i,j}$, $\delta\Pi_{ij}$, etc;

2) to find solutions of the equations that describe the time evolution of these variables in the perturbed state.

To solve the first problem, it is sufficient to note that

$$\begin{aligned}\langle\psi_0'|O|\psi_0'\rangle&=\int d\mathbf{r}\int d\mathbf{r}'\rho(\mathbf{r},\mathbf{r}',t)O(\mathbf{r}',\mathbf{r})\\&=\int d\mathbf{r}\int d\mathbf{p}(\rho O)_W\\&=\int d\mathbf{r}\int d\mathbf{p}\exp\left\{\frac{\hbar}{2i}(\nabla_{\mathbf{r}}^0\cdot\nabla_{\mathbf{p}}^f-\nabla_{\mathbf{p}}^0\cdot\nabla_{\mathbf{r}}^f)\right\}O_W(\mathbf{r},\mathbf{p})f(\mathbf{r},\mathbf{p},t),\end{aligned}$$

if O is a single-particle operator. Here, we have used Eq. (4), and O_W is the Wigner transform of the operator O.

To solve the second problem, it is necessary to add the potential $O(t)$ to the Hamiltonian in Eq. (1). Then on the right-hand side of Eq. (5) there appears the term

$$\frac{2}{\hbar}\sin\left\{\frac{\hbar}{2}(\nabla_{\mathbf{r}}^0\cdot\nabla_{\mathbf{p}}^f-\nabla_{\mathbf{p}}^0\cdot\nabla_{\mathbf{r}}^f)\right\}\times[O_W(\mathbf{r},\mathbf{p})f(\mathbf{r},\mathbf{p},t)e^{-i\omega t}+\text{c.c.}]. \tag{46}$$

Making the same calculations as before, we obtain equations for all the moments of the Wigner function needed to calculate $\langle\psi_0'|O|\psi_0'\rangle$. The only new element will be the presence of the term (46), which makes the equations for the moments inhomogeneous.

For 2^+ excitations, it is necessary to take as perturbation the operator of the quadrupole electric moment:

$$\hat{q}_{2\mu}=(Z/A)e_p r^2 Y_{2\mu}.$$

Its Wigner transform $q_{2\mu}(\mathbf{r})$ is given by the same expression, and for the matrix element we have

$$\begin{aligned}\langle\psi_0'|\hat{q}_{2\mu}|\psi_0'\rangle&=\int d\mathbf{r}\int d\mathbf{p}q_{2\mu}(\mathbf{r})f(\mathbf{r},\mathbf{p},t)\\&=\int d\mathbf{r}n(\mathbf{r},t)q_{2\mu}(\mathbf{r})\\&=Q_{2\mu}^{(0)}+\delta Q_{2\mu}(t).\end{aligned}$$

Solving the inhomogeneous equations corresponding to the system (43), we can find all the $\delta Q_{2\mu}$, and, by means of (45), the reduced transition probabilities. For example, for the γ mode we obtain

$$|\langle\psi_{\gamma_1}|\hat{q}_{2+2}|\psi_0\rangle|^2=\frac{3}{2\pi}\frac{\hbar e_p^2Z^2}{mA}\frac{a_1^2(\omega_1+4\Omega)}{(\omega_1-\omega_2)(\omega_1-\omega_3)}.$$

Here, ω_i are the roots of the equation $\mathcal{F}_\gamma(\omega,-\Omega)=0$. Naturally, the factor $(\omega_1+4\Omega)/(\omega_1-\omega_2)(\omega_1-\omega_3)$ must be positive. This requirement gives rise to selection rules. Thus, it is readily seen that this factor is negative for the root of smallest absolute magnitude, since the two other roots (larger in absolute magnitude) always have opposite signs. Therefore, the lowest γ mode cannot be excited by the operator $\hat{q}_{2+2}$! It is easy to show that it is excited by the operator $\hat{q}_{2-2}$, i.e., with a decrease of angular momentum by $2\hbar$, as is reflected in Fig. 3, which shows pieces of the spectrum of collective excited states of spheroids at two characteristic values of I. At the lower spin, the unexcited state belongs to an yrast sequence; at the higher spin, to a sequence of stationary axial configurations that are not yrast configurations of the nuclei. The second case is noted in Fig. 3 ("yrast–axial").

It can be seen from Fig. 3 that, owing to the presence of soft modes, it is possible at all values of I for transitions to occur that lead to excitation of the γ mode and reduction of the angular momentum by two Planck units, the energies of such transitions being $E_\gamma\simeq 2\hbar\Omega$, i.e., very close to the energy of transitions along the yrast line.

The second low-lying branch (α mode) also makes it possible to realize transitions practically along the yrast line, but in this case with $E_\gamma\simeq 1\hbar\Omega$ and with $\Delta I=1\hbar$. The possibility of such transitions in connection with just such low-energy wobbling modes of nonaxial nuclei was discussed in Ref. 55.

There is quite good experimental support for the existence of low-lying modes. Figure 4, which is taken from Refs. 56 and 57, shows the spectrum of γ rays that deexcite high-spin states of the ^{154}Er nucleus produced in the ^{40}Ar + ^{118}Sn reaction after the emission of four neutrons. Analysis of the anisotropy of the radiation shows that the region of the peak with higher energy is formed almost exclusively by quadrupole transitions with $\Delta I=2\hbar$ possessing a high degree of collectivity (about 140 Weisskopf units), in quite good agreement with the properties of the low-lying γ mode.

In the region of the lower peak, it is necessary to allow a significant contribution of transitions with $\Delta I=1\hbar$. The intensity of the lower peak increases rapidly with increasing angular momentum, suggesting that the effect could be sustained by the precession mode. Finally, the ratio of the energies at which the peaks are observed agrees with the theoretical ratio: $E_\gamma/E_\alpha\simeq 2$.

Much interesting information is supplied by analysis of the model-independent energy-weighted sum rule[58]

$$\sum_i(E_i-E_0)|\langle i|F|0\rangle|^2=\frac{1}{2}\langle 0|[F,[H,F]]|0\rangle.$$

In our case, F is the operator of the quadrupole moment. Calculations show that the single 2^+ level (for $\Omega=0$) that we identified with the giant quadrupole resonance completely exhausts the sum rule. This is not surprising; for the equations of motion were from the very start de-

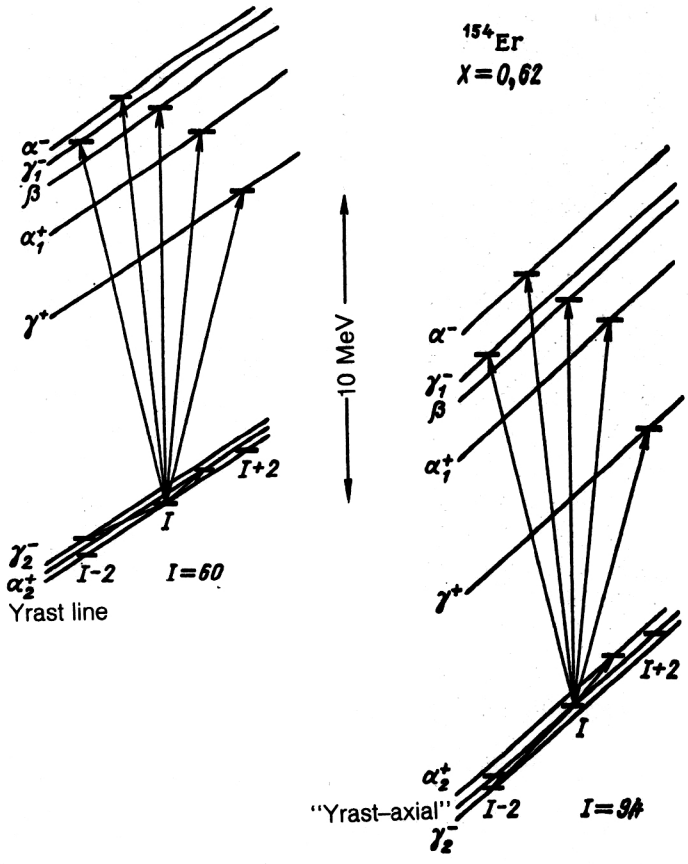

FIG. 3. Allowed electromagnetic transitions between equilibrium and excited states for two characteristic values of the rotation velocity for ^{154}Er ($X = 0.62$); spheroids.

rived for multipole moments, so that the excitations described by them must exhaust the corresponding sum rule to 100%.

If approximations were made in their derivation, then only in the potential, and this should not affect the sum rule (provided the potential does not depend on the momentum). But what does this fact of 100% exhaustion mean? The answer suggests itself: In writing down and solving the equations of motion for the quadrupole moment, we find the positions of the centroids of all possible 2^+ states in the nucleus. When we determine the displacement vector in more detail [continue the series (31)], and also take into account other degrees of freedom (spin, isospin), the centroid will be split into a larger number of 2^+ levels. For example, allowance for the isospin degrees of freedom leads to splitting into isoscalar and isovector resonances with approximately equal strengths without significantly changing the position of the centroid of the isoscalar giant quadrupole resonance.

Here, the following fact should be noted. In the case of ellipsoidal configurations, the modes shown in Fig. 2 do not exhaust the sum rule! The fact is that the system of equations (43) admits not only solutions of vibrational type ($\sim e^{i\omega t}$) but also nonvibrational solutions; a detailed analysis of them and a derivation of the corresponding spectra and mass parameters can be found in Ref. 26.

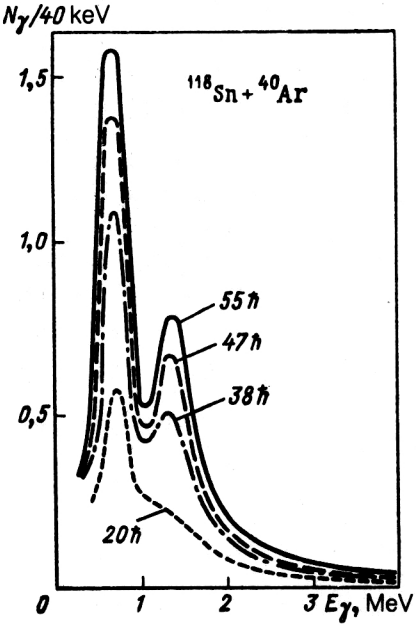

FIG. 4. Spectra of γ rays for some initial spin values of the states from which a cascade begins.

Spreading widths of quadrupole excitations of rotating nuclei

To take into account the spreading width of resonances in the framework of our approach, it is necessary to modify the equation for the density matrix [see Eq. (1)], by adding a so-called correlation term[4] on the right-hand side. The Wigner transform of the correlation term can be called a

collision integral in view of the manifest analogy between Eq. (6) and the Boltzmann–Vlasov equation.[59]

The collision integral of classical kinetic theory conserves the density, momentum, and kinetic energy of the matter at each point of space. The collision integral is given analogous properties in nuclear models in view of the short range of the nuclear forces. Thus, we assume that

$$\int d\mathbf{p} I(f)=0,$$

$$\int d\mathbf{p} I(f) p_i=0, \quad i=1,2,3,$$

$$\sum_{s=1}^{3} \int d\mathbf{p} I(f) p_s^2=0. \tag{47}$$

Taking into account (47), we can show that Eqs. (8) and (9) are not changed at all when the collision integral is included. Equations (10) and (11) for the tensor $P_{i_1 \ldots i_n}$ acquire a right-hand side equal to

$$m^{1-n} \int I(f)(p_{i_1}-mu_{i_1}) \ldots (p_{i_n}-mu_{i_n}) d\mathbf{p}.$$

A convenient (in the sense of application of the moment method) representation of $I(f)$ is the expansion

$$I(f)=a(f-f^{(0)})+b_0 f^{(0)}(\mathbf{r},\mathbf{p}) + \sum_{[n]} b_{[n]} \frac{\partial^n}{\partial p_{i_1} \ldots \partial p_{i_n}} f^{(0)}(\mathbf{r},\mathbf{p}),$$

where $f^{(0)}(\mathbf{r}, \mathbf{p})$ is the distribution function for the stationary state, and the sum is taken over all possible sets of indices subject to $n \geqslant 1$. In classical theory, the coefficients a, b_0, $b_{[n]}$ are functions of r and t, while in the quantum case they are functionals of $\delta f \equiv f - f^{(0)}$, containing information about the entire "prehistory" of the system.

For such a parametrization of the collision integral, Eqs. (47) are satisfied if the following relations hold:

$$b_0=-a\delta n/n_0, \quad b_i=ma\delta u_i,$$

$$\sum_{i=1}^{3} b_{ii}=\frac{am}{2n_0} \sum_{i=1}^{3} \left(\frac{\delta n}{n_0} P_{ii}^{(0)} - \delta P_{ii} \right).$$

Here, we have ignored the terms quadratic in the variations. We analyze the dependence on the angular momentum of the width of the collective excitations of quadrupole type under the assumption that $a=-1/\tau=\text{const}$, $b_{ij} = \delta_{ij}(am/6n_0)\Sigma_{s=1}^{3}((\delta n/n_0)P_{ss}^{(0)} - \delta P_{ss})$, taking all the remaining functions $b_{[n]} = 0$. This approximation is a corrected (with respect to conservation of the matter density and the momentum and kinetic energy of the colliding particles) form of the "relaxation time" approximation, which is popular in physical kinetics[30,60] and is often used in nuclear models.[60–62] We regard τ as a parameter which we determine in such a way as to reproduce the widths of the giant quadrupole resonance observed experimentally in the absence of rotation.

The vibrational solutions of the transformed system (43) become complex. We identify the real part of the frequency with the excitation energy $E=\hbar \operatorname{Re} \omega$, and the imaginary part with its width (Ref. 63): $\Gamma=2\hbar \operatorname{Im} \omega$.

The calculations of Ref. 64 show that there is no clear correlation between the widths of the individual branches of the resonance and their energies. However, the widths of the low-lying vibrational modes are appreciably less than the widths of the branches that represent the giant quadrupole resonance. The relation $\Gamma \ll \Delta E_\gamma$, where ΔE_γ is the energy carried away by the γ rays in a collective transition that de-excites or excites a vibrational mode, is satisfied for all branches of the spectrum of quadrupole vibrations.

It should be noted that already at the bifurcation point, i.e., at $I \simeq 70\hbar$ (for ^{154}Er) the spheroidal configuration becomes unstable with respect to quadrupole vibrations (the width of the low-lying γ mode becomes negative).

The ellipsoidal configurations are stable with respect to excitations of quadrupole symmetry in the entire range of angular momenta allowed by the conditions (40) of secular equilibrium.

Nuclei with static deformation

We consider nonrotating nuclei possessing static deformation.[65]

We begin with an analysis of the conditions for equilibrium. They can be readily obtained from (34) by setting $\Omega=0$ and $i=j$:

$$2\sigma_{ii} - \mathscr{U}_0 + C_{ii} - \Pi_{ii}=0, \quad i=1,2,3. \tag{48}$$

In the case of a spherical nucleus ($\delta=0$), all three equations (48) are the same:

$$\Pi_0 - 2\sigma(0) - C(0) + \mathscr{U}_0=0. \tag{49}$$

Here $\sigma(0) = \frac{2}{3}\pi TR^2$, $C(0) = -\frac{4}{3}X\pi TR^2$.

Subtracting Eq. (49) from (48), we obtain

$$\Pi_{ii}=2(\sigma_{ii}-\sigma(0)) + (C_{ii}-C(0)) + \Pi_0. \tag{50}$$

This relation demonstrates very well the part played by the various tensors in the appearance of static deformation of nuclei. First, we note the absence of a tensor of volume nuclear energy. Therefore, as we have already noted, it has no significance for the description of nuclear shapes. Moreover, one can show that Coulomb and nuclear surface forces are also not decisive for the appearance of static deformation. Indeed, let us consider the difference of any two equations in (50):

$$\Pi_{11} - \Pi_{33}=2(\sigma_{11}-\sigma_{33}) + (C_{11}-C_{33}). \tag{51}$$

If the pressure tensor is assumed to be isotropic, i.e., $\Pi_{11} = \Pi_{22} = \Pi_{33}$, then (51) reduces to

$$2(\sigma_{11}-\sigma_{33}) + (C_{11}-C_{33})=0.$$

Earlier (see Sec. 1), it was shown that besides the trivial solution $\delta=0$ this equation also has solutions with $\delta \neq 0$. However, they are possible only for very exotic values of the fissility parameter: $X>1$ for oblate spheroids and $X>0.89$ for prolate spheroids. Thus, for practically all existing nuclei the assumption of isotropy of the pressure

tensor leads unambiguously to a spherical shape. Deformation is possible only for an anisotropic pressure tensor.

Using some realistic model of the ground state (say, the Nilsson scheme), one could calculate Π_{ii}, and then, from (51), determine the deformation of the nucleus. However, such work goes beyond the scope of this review, and therefore we shall take δ from the experiments and determine the dependence of Π_{ii} on the deformation from (50).

Small-amplitude quadrupole vibrations are described by the system of equations (41) with $\Omega = 0$. A calculation shows that the giant quadrupole resonance is split into three branches, with each of which one can associate a definite modulus $|\mu|$ of the projection of the multipole moment.

The splitting of the giant quadrupole resonance by the deformation is manifested experimentally in a broadening of it. The broadening obtained in our calculations changes little from nucleus to nucleus (see Table I) and is on the average 1 MeV, in reasonably good agreement with both model[66] and realistic[67,68] calculations in the RPA, and also with calculations based on equations of hydrodynamic type.[69]

We determine the experimental broadening simply, by subtracting from the width of the giant quadrupole resonance of the deformed nucleus the width of the same resonance of the corresponding spherical isotope. The data of different authors[70–72] contain a rather large spread (see Table I), but nevertheless the greater part of the theory agrees with experiment. In the study of Ref. 73, it was possible to measure directly the distance between the γ and β branches in ^{152}Sm: $\Delta E = 2.1 \pm 0.6$ MeV, and this is also not too far from the theoretical value 1.2 MeV.

3. VIRIAL EQUATIONS FOR SECOND-RANK TENSORS. REALISTIC NONLOCAL INTERACTION

In this section, we shall demonstrate the possibilities of the method of moments when one is working with a realistic nucleon–nucleon interaction and realistic matter distribution in the nucleus (diffuse edge).[74] As a realistic interaction, we make use of the variants of Skyrme forces: SKM*.[75] A self-consistent potential is obtained from the forces in the usual manner;[76] we take it from Ref. 45. The realistic interaction distinguishes protons and neutrons, and therefore the density matrix acquires a corresponding index τ, and Eq. (1) is slightly modified:[76]

$$i\hbar \frac{\partial \hat{\rho}_\tau}{\partial t} = [\hat{H}_\tau \hat{\rho}_\tau]. \tag{52}$$

Here, the Hamiltonian has the form

$$\hat{H}_\tau = -\nabla \frac{\hbar^2}{2m_\tau^*} \nabla + U_\tau(r) + \frac{1}{2i}(\nabla \cdot I_\tau + I_\tau \cdot \nabla),$$

where

$$\frac{1}{m_\tau^*} = \frac{1}{m} + 2C_\tau(\mathbf{r}),$$

$$C_\tau = \frac{1}{4\hbar^2}\left(t_+ n - \frac{t_-}{2} n_\tau\right), \quad n = n_\tau + n_{\tau'},$$

$$U_\tau = t_0\left[(1 + \chi_0/2)n - \left(\chi_0 + \frac{1}{2}\right) n_\tau\right] + \frac{1}{16}(3t_2 - 5t_1)\nabla^2 n + \frac{1}{32}(3t_2 + 5t_1)\nabla^2 n_\tau + \frac{1}{2}\eta[P + m(n_\tau u_\tau^2 + n_{\tau'} u_{\tau'}^2)] - \frac{mt_-}{8\hbar^2}(P_\tau + m n_\tau u_\tau^2) + \delta_{\tau,p} U_c + \frac{t_3}{24} n^{\sigma-1}\{n^2[(1-\chi_3)\sigma + 2(2 + \chi_3)] - 2n_\tau^2 \sigma(1 + 2\chi_3) + 2nn_\tau(\sigma - 1)(1 + 2\chi_3)\},$$

$$I_\tau = -\frac{t_+}{2} j + \frac{t_-}{4} j_\tau, \quad j_\tau = n_\tau u_\tau,$$

TABLE I. Broadening of the giant quadrupole resonance due to the deformation $\delta = 3\varepsilon/[2(3-\varepsilon)]$ (Ref. 3). (References are given in square brackets.)

Element	Deformation (Ref. 3)	Width of GQR_{exp}, MeV	Broadening of GQR_{exp}, MeV	Broadening of GQR_{theor}, MeV
^{96}Mo	0	4,8±0,6 [70]	—	—
^{100}Mo	0,19	5,1±0,5 [70]	0,3	1,0
^{142}Nd	0	3,8 [70]	—	—
^{150}Nd	0,23	5,0±0,2 [70]	1,2±0,2	1,0
^{144}Sm	0	3,9±0,2 [71] 5,4 [70] 2,4±0,2 [72]	—	—
^{152}Sm	0,26	4,3±0,2 [71]	0,4±0,3	1,2
^{154}Sm	0,28	4,7±0,3 [71] 5,5 [70] 3,7±0,3 [72]	0,8±0,3 0,1 1,3±0,3	1,3
^{164}Dy	0,28	—	—	1,3
^{174}Hf	0,25	—	—	1,1
^{230}U	0,22	—	—	0,9

$U_c(\mathbf{r}) = \int n_p(\mathbf{r}')(e_p^2/|\mathbf{r} - \mathbf{r}'|)d\mathbf{r}'$ is the direct part of the Coulomb potential, $P_\tau = \Sigma_{s=1}^{3} P_{s\tau}$, $P = P_\tau + P_{\tau'}$, $t_+ = t_1 + t_2$, $t_- = t_1 - t_2$, $\eta = mt_+/2\hbar^2$, $\sigma, t_0, t_1, t_2, t_3, \chi_0, \chi_3$ are the SKM* parameters, $\tau' = p$ (protons) if $\tau = n$ (neutrons), and vice versa.

Performing on Eq. (52) the operations described in Sec. 1, we obtain the modified continuity equation

$$\frac{\partial n_\tau}{\partial t} + \operatorname{div}(n_\tau u_\tau) = \eta \operatorname{div}[n_\tau n_{\tau'}(u_{\tau'} - u_\tau)], \qquad (53)$$

an (Euler) equation of motion, and an equation for the stress tensor.

Further, in accordance with the prescriptions of Sec. 1, we must integrate these equations over $\mathbf{r}$ with corresponding weights and investigate the variations of the obtained virial equations, and also their static solutions. Here we must mention several new elements.

First, on the right-hand side of the continuity equation (53) a term that imitates exchange effects has appeared.[76] Accordingly, the formula for the variation of the density is changed:

$$\delta n_\tau = -\operatorname{div}(n_\tau \xi_\tau) - \eta \operatorname{div}[n_\tau n_{\tau'}(\xi_\tau - \xi_{\tau'})].$$

Second, because of the nonlocality of the Skyrme forces, terms containing the tensors P_{ijk} remain in the virial equations. They must be ignored. However, by means of sum rules one can show that the importance of the discarded terms is negligible.

Finally, because the interaction is realistic, it is possible to study both isoscalar (*IS*) and isovector (*IV*) excitations, and also compression modes. Details of the calculations can be found in Ref. 74.

The static solution of the virial equations gives a relationship between the shape parameters of the nucleus and the rotation velocity that almost exactly reproduces the results of Sec. 2. At the same time, it turns out that the nonlocal part of the Skyrme forces (the coefficients t_+ and t_-) correspond to surface tension.

The vibrational solutions of these equations describe 2^+, 1^+, and 0^+ excitations. For $\Omega = 0$, the theory gives *IS* and *IV* 2^+ and 0^+ levels—giant resonances. The theoretical values of the isoscalar giant quadrupole and monopole resonances practically agree with the experiments. The situation with regard to the isovector giant quadrupole and monopole resonances is not so brilliant—the degree of agreement between theory and experiment can be gauged from Figs. 5 and 6. Here, we have also verified the accuracy of the approximation, widely used in such calculations, in which it is assumed that the neutrons and protons move in phase ($V_{i,jn}/Z_n = V_{i,jp}/Z_p$ for *IS* vibrations) or in antiphase ($V_{i,jn}/Z_n = -V_{i,jp}/Z_p$ for *IV* vibrations). The approximate solution practically coincides with the exact solution for the *IS* modes and differs appreciably from the exact solution for the *IV* modes—in heavy nuclei, the error reaches 10%. The calculations show that the *IV* and *IS* modes make almost equal contributions to the electromagnetic energy-weighted sum rule and completely exhaust it, each of them, in their turn, almost completely exhausting the corresponding *IV* or *IS* sum rule.

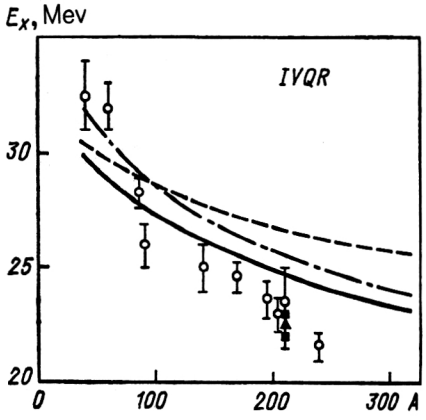

FIG. 5. Calculated and experimental energies of the isovector quadrupole resonance (IVQR) as a function of the atomic mass. The continuous curve gives the equilibirum density parametrized by Bernstein;[78] the chain curve gives the density according to Bohr and Mottelson;[77] the broken curve represents the approximate calculation. The experimental data are taken from Ref. 79.

For $\Omega \neq 0$, both giant quadrupole resonances split into five branches, and five low-lying modes also appear (Fig. 7). Two of them (5 and 12, the *IS* γ and α modes) were already described in Sec. 2. They have now acquired isovector analogs—curves 6 and 11).

The fifth low-lying mode (curve 13) corresponds to vibrations of the proton angular momentum relative to the neutron angular momentum (like scissors). This mode does not have an isoscalar analog, since the total angular momentum is conserved. This mode can be classified as an isovector 1^+ excitation. In nuclei possessing static deformation, it is known as the scissors mode, or angular resonance.[81–83]

Mode 11 can also be interpreted as a kind of rotational motion of the neutrons relative to the protons. Indeed, it was shown in Sec. 2 that its isoscalar analog (mode 12) has much in common with precession. Therefore, the corresponding isovector mode (11) must describe "precession" of the proton matter relative to the neutron matter.

Thus, we have two 1^+ levels of magnetic type. If, following Ref. 84, we take into account the possible deviation of the nuclei from axial symmetry, then these levels must be split. Allowance for the spin degrees of freedom, and also for deformation of the Fermi surface of higher multipolarity, can lead to a further splitting of them. Other authors, using various phenomenological approaches,[85–89] obtain, as a rule, one or two levels. In the more realistic calculations of the RPA + HFB type the scissors mode is strongly fragmented.[90–94] In the first experiments that confirmed the existence of such a mode,[95] one level was obtained at $E_{1+}(\text{exp}) = 3.1$ MeV (^{156}Gd) with $B(M1)\uparrow_{\text{exp}} = (1.3 \pm 0.2)\mu_N^2$. As the experimental techniques were improved, the number of levels increased, so that in the recent study of Ref. 96 the number of levels observed in this nucleus was already ten, with energies from 2.027 to 3.218 MeV and with $B(M1)\uparrow$ from $0.09\mu_N^2$ to $1.21\mu_N^2$, these satisfying $\Sigma_{i=1}^{10} B_i(M1)\uparrow = 2.66\mu_N^2$.

If we ignore the fact that in the given study the nuclei

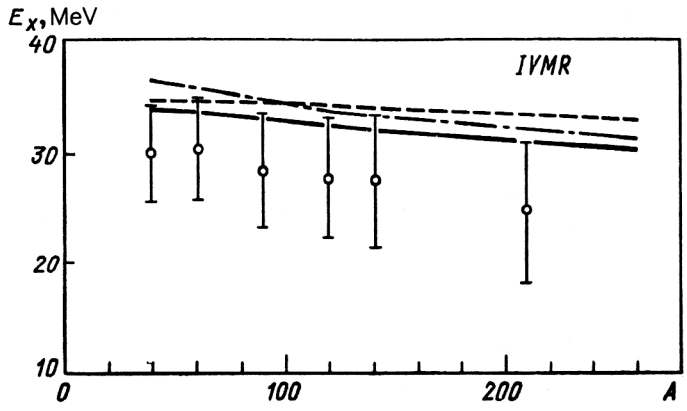

FIG. 6. Calculated and experimental values of the energy of the isovector monopole resonance (IVMR) as functions of the atomic mass. The experimental data are taken from Ref. 80. See also the explanations to Fig. 5.

were deformed only because of rotation and we compare the energy of 1^+ excitations calculated here for the corresponding deformation $\delta = 0.25$ ($e \simeq 0.66$, $I \simeq 69\hbar$; see Fig. 7) with the experimental values for the nucleus ^{156}Gd, then

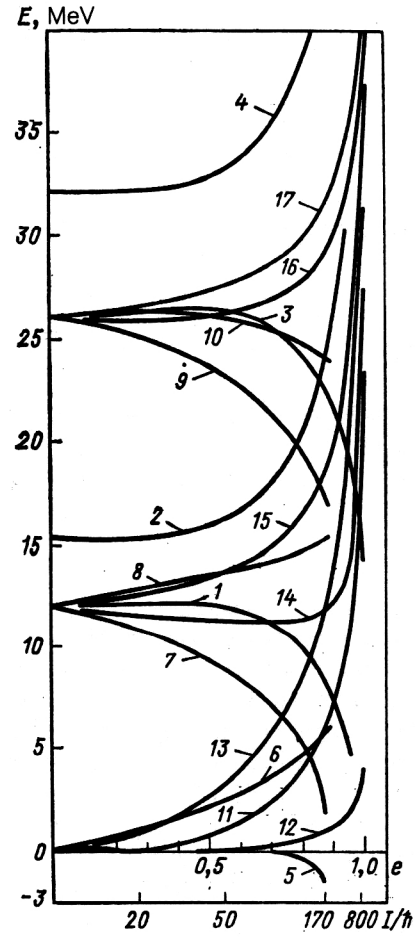

FIG. 7. Energies of 0^+, 1^+, and 2^+ excitations as functions of the eccentricity $e = (1 - a_3^2/a_1^2)^{1/2}$ and angular momentum I. The levels are labeled as follows. Isoscalar: 1 (0), 2 (0), 5 (− 2), 7 (+ 2), 8 (− 2), 13 (− 1), 14 (+ 1), 15 (− 1); isovector: 3(0), 4(0), 6 (− 2), 9 (+ 2), 10 (− 2), 16 (+ 1), 17 (− 1) (the value of μ, the projection of the corresponding multipole moment onto the rotation axis, is given in brackets). The isoscalar level 12 and the isovector level 11 have projection $\mu = -1$ up to the point at which the energy becomes zero and the value $\mu = +1$ after this point.

we find that the theory gives the entirely reasonable values $E_{1+}(13) \simeq 4.9$ MeV and $E_{1+}(11) \simeq 1.9$ MeV. The theoretical values of the $B(M1)$ factors of the two levels are practically the same ($\simeq 1.6\mu_N^2$), and this is also in reasonable agreement with experiment.

We note also that all the experimental studies confirm the orbital nature of the mode under discussion, in complete correspondence with our theory. Levels 12 and 11, which disappear at $I_c \simeq 25\hbar$ and $I_c \simeq 19\hbar$, respectively, have an interesting feature—at the corresponding point, the sign of μ for them changes from minus to plus. This means that the modes corresponding to these levels are excited with a decrease of the spin at $I < I_c$ and with an increase of the spin at $I > I_c$. The mechanical interpretation of this phenomenon is also of some interest. Since mode 12 is related to a precession mode, one can say that at the point I_c the direction of the precession vector changes.

Calculation of the $B(E2)$ factors reveals a high collectivity (15–20 Weisskopf units) of the high-lying modes and a significant increase in the collectivity of the low-lying modes (especially modes 5 and 12) with increasing I. For the isoscalar excitations, the $B(E2)$ factors are, as a rule, greater than for the isovector excitations.

Figure 8 shows the calculated $B(E0)$ factors as functions of e and I. It can be seen that in the region $e \simeq 0.6$–0.7 ($\delta \simeq 0.3$) the $B(E0)$ factors of the breathing mode and of the β mode become equal in magnitude. This indicates strong mixing of the monopole and quadrupole excitations in the deformed nuclei. Experimentally, it is manifested in a "splitting" of the giant monopole resonance in deformed nuclei.[97,98] It is interesting that this result can be reproduced qualitatively in a rather simple schematic model.[66]

4. VIRIAL EQUATIONS FOR THIRD-RANK TENSORS. NEGATIVE-PARITY EXCITATIONS

In this section, we study the dynamics of third-rank tensors and, accordingly, the properties of negative-parity excitations.

Basic equations

In the case of an equilibrium state, the virial equations (21)–(23) are satisfied identically and do not give any useful information. The small-amplitude motions are de-

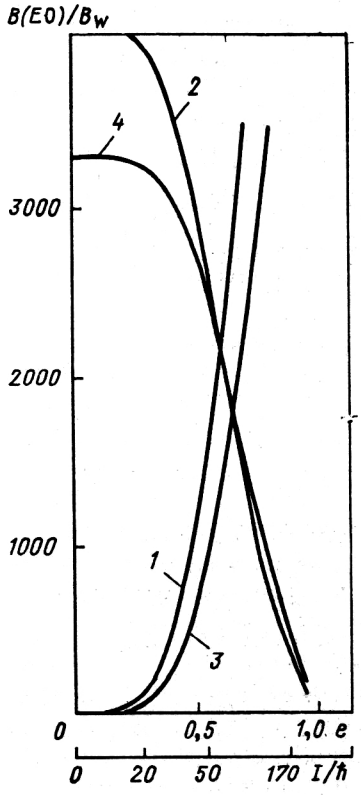

FIG. 8. The $B(E0)$ factors of transitions from excited states to the ground state. The curves are numbered in accordance with the numbering of the levels (see Fig. 7) from which the transitions take place.

scribed by the system of equations (30). In the expansion (31) for $\xi_i(\mathbf{r},t)$, we restrict ourselves to the first three terms of the series, and in the expansion (32) for $\delta P_{ij}(\mathbf{r},t)$ to two terms.

In accordance with Sec. 1, the dynamics of the third-rank Cartesian tensors must be studied simultaneously with the dynamics of the second-rank tensors. If protons and neutrons are not distinguished, then the only first-rank tensor will be the coordinate $V_i(t)$ of the center of mass of the nucleus, which is fixed by the condition

$$V_i(t)\equiv\int \xi_i(\mathbf{r},t)\rho(\mathbf{r},t)d\mathbf{r}=0.$$

The potential-energy tensor present in (30), $\mathscr{W}_{ai,jk}$, is a sum of two terms:

$$\mathscr{W}_{i,jk}=\mathscr{U}_{i,jk}+C_{i,jk}.$$

The Coulomb part has the form

$$C_{i,jk}=\left(\frac{Ze_p}{A}\right)^2\int n(\mathbf{r})x_jx_k\frac{\partial}{\partial x_i}\int\frac{n(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'d\mathbf{r}.$$

In the approximation of constant density and a sharp edge of the nucleus, it is helpful to decompose the tensor of the nuclear forces into volume and surface parts:

$$\mathscr{U}_{i,jk}=\int_V nx_jx_k\frac{\partial U}{\partial x_i}d\mathbf{r}$$

$$=n_0\int_V x_jx_k\frac{\partial U}{\partial x_i}d\mathbf{r}$$

$$=\Sigma_{i,jk}-\delta_{ij}\mathscr{U}_k-\delta_{ik}\mathscr{U}_j,$$

where $\Sigma_{i,jk}=n_0\int_S Ux_jx_ks_id\mathbf{S}$, $\mathscr{U}_j=n_0\int_V x_jUd\mathbf{r}$.

We parametrize the average field U on the surface of the nucleus by a surface tension [see Eq. (38)]. For ellipsoids, the tensors $\Sigma_{i,jk}$ and $C_{i,jk}$ obviously vanish. Expressions for the variations $\delta\Sigma_{i,jk}$ were obtained in Ref. 49, and for $\delta C_{i,jk}$ in Ref. 16. Naturally, they are combinations of $V_{i,jk}$ and of multiple-index symbols.

It is necessary to dwell especially on the problem of calculating the quantum correction $\delta\chi_{ijk}$.

In the case of a self-consistent average field, its variation is given by

$$\delta W(\mathbf{r})=\int v(\mathbf{r},\mathbf{r}')\delta n(\mathbf{r}')d\mathbf{r}'.$$

By means of it, we obtain

$$\delta\chi_{ijk}=\int\int d\mathbf{r}'\,d\mathbf{r}\left\{\delta n(\mathbf{r})n(\mathbf{r}')\frac{\partial^3v(\mathbf{r},\mathbf{r}')}{\partial x_i\partial x_j\partial x_k}+n(\mathbf{r})\delta n(\mathbf{r}')\frac{\partial^3v(\mathbf{r},\mathbf{r}')}{\partial x_i\partial x_j\partial x_k}\right\}$$

$$=\int\int d\mathbf{r}'\,d\mathbf{r}n(\mathbf{r}')\delta n(\mathbf{r})\left\{\frac{\partial^3v(\mathbf{r},\mathbf{r}')}{\partial x_i\partial x_j\partial x_k}+\frac{\partial^3v(\mathbf{r},\mathbf{r}')}{\partial x_i'\partial x_j'\partial x_k'}\right\}.$$

If the conditions of Galilean invariance are satisfied, i.e., $v(\mathbf{r},\mathbf{r}')=v(|\mathbf{r}-\mathbf{r}'|)$, then

$$\frac{\partial v}{\partial x_i}=-\frac{\partial v}{\partial x_i'}$$

and the quantum correction $\delta\chi_{ijk}$ vanishes.

But if the average field is not self-consistent, then the quantum correction begins to depend on the form of the potential. Thus, in our case, in which the average field on the surface of the nucleus is approximated by surface tension, for the quantum correction we obtain[27,28]

$$\delta\chi_{ijk}=2\pi T\frac{\hbar^2}{m^2}\left\{\sum_{s=1}^{3}\left[\delta_{ik}\left(\frac{1}{5}L_{j,ss}-\frac{1}{7}L_{jss}\right)+\delta_{ij}\left(\frac{1}{5}L_{k,ss}-\frac{1}{7}L_{kss}\right)+\delta_{kj}\left(\frac{1}{5}L_{i,ss}-\frac{1}{7}L_{iss}\right)\right]-\frac{2}{7}L_{ijk}\right\},$$

where $L_{ijk}=L_{i,jk}+L_{j,ki}+L_{k,ij}$.

With allowance for all these remarks, the system of equations (30) takes the form

$$\ddot{V}_{i,jk}+\delta C_{i,jk}+\delta\Sigma_{i,jk}-\delta_{ij}\delta\mathscr{U}_k-\delta_{ik}\delta\mathscr{U}_j-\delta\Pi^k_{ij}-\delta\Pi^j_{ik}=0;$$

$$\delta\dot{\Pi}^k_{ij}+\frac{2A}{9m}\langle p^2r^2\rangle(\dot{L}_{i,jk}+\dot{L}_{j,ik})-\delta\Pi_{ijk}=0;$$

$$\delta\dot{\Pi}_{ijk}-\frac{\langle p^2\rangle}{m^2\langle r^2\rangle}\left[\sum_{s=1}^{3}(\delta_{jk}\delta\Pi^s_{is}+\delta_{ik}\delta\Pi^s_{js}+\delta_{ij}\delta\Pi^s_{ks})-\delta\Pi^k_{ij}-\delta\Pi^i_{jk}-\delta\Pi^j_{ki}\right]=\delta\chi_{ijk}. \quad (54)$$

The 46 equations of this system must be recombined to make them equations for the irreducible tensors of multipolarity $\lambda = 3, 2, 1$. The corresponding systems of equations can be found in Ref. 33.

3^- excitations

The theory gives two 3^- levels. In the simplest approximation, they can be written as

$$E^{(h)}_{3^-}=\hbar\sqrt{\frac{43}{5}}\frac{v_F}{R}\simeq 130A^{-1/3}\ \text{MeV}, \quad (55)$$

$$E^{(l)}_{3^-}=\hbar\sqrt{\frac{15}{43}\left(\frac{7}{4}-X\right)\frac{160\pi}{7mA}\,T}$$

$$\simeq 31.4\sqrt{\frac{7}{4}-X}A^{1/2}\ \text{MeV}.$$

It is obvious that $E^{(h)}_{3^-}$ is the centroid of the giant octupole resonance, while $E^{(l)}_{3^-}$ can be interpreted as the centroid of all the 3^- levels lying below the giant octupole resonance.

It is here appropriate to discuss a widely used approximation with which we have been able to dispense. This is the crude truncation of the chain of p moments of Eq. (6) at Eq. (10), in which the tensors P_{ijk}, P_{ijkl}, etc., are ignored in the hope that with increasing rank of the tensors $P_{i...l}$ their influence decreases rapidly, i.e., it is hoped that there exists a certain "convergence with respect to the rank." If, following this logic, we set $P_{ijk} = 0$ and ignore the last term in (54), then we obtain only one 3^- state with energy

$$\hbar\omega_{3^-}(P_{ijk}=0)=\hbar\frac{v_F}{R}\sqrt{\frac{28}{5}}\simeq 104.5A^{-1/3}\ \text{MeV},$$

which agrees exactly with the result of Nix[12] but differs too much from (55) for one to be able to speak of any convergence.

It can be seen from analysis of the relations between the amplitudes that the octupole deformation of the Fermi surface is equally important for the high- and low-lying excitations, while the effects of the quadrupole deformation of the Fermi surface are manifested effectively in only the high-lying excitation. The distribution of the collective currents is the same for the two modes.

The calculations show that the giant octupole resonance exhausts about 70% of the energy-weighted sum rule, and $E^{(l)}_{3^-}$, accordingly, about 30%, in qualitative agreement with the experimental data. We note also that allowance for the quantum correction has practically no influence on the energy of the 3^- excitation but is important for the calculation of the $B(E3)$ factors of the low-lying 3^- states, bringing the results closer to the experimental values by $\sim 10\%$ in heavy nuclei and $\sim 20\%$ in light ones.

2^- and 1^- excitations

Besides the two 3^- levels, Eqs. (54) also give one 2^- level and one 1^- level. In the simplest approximations, we can represent the energy of the 2^- excitation, which we identify with the magnetic quadrupole resonance, in the form

$$E_{2^-}\simeq\hbar\sqrt{\frac{7}{5}}\frac{v_F}{R}\simeq 52.3A^{-1/3}\ \text{MeV}.$$

This formula describes the experimental data quite well. Calculations of the $B(M2)$ factors show that they do not differ in any nucleus from the experimental values by more than a factor of three; this can be regarded as reasonable agreement, since we do not take into account the spin degrees of freedom.

The isoscalar 1^- excitation corresponds to vibrations of the toroidal dipole moment[99] of the nucleus: $\mathbf{T} = (e_pZ/10cA)\int n[\mathbf{r}(\mathbf{r}\cdot\mathbf{u}) - 2r^2\mathbf{u}]d\mathbf{r}$. Its energy is given by

$$E_{1^-}=\hbar\sqrt{\frac{11}{5}}\frac{v_F}{R}\simeq 65.6A^{-1/3}\ \text{MeV},$$

which is close to the result of Ref. 100. At the present time, there is no sufficiently convincing experimental information about such excitations, although there are some speculations on this subject.[101,102]

3^- and 2^- excitations with a realistic nonlocal interaction

We demonstrate the results of calculations of the energies of the 3^- and 2^- excitations with Skyrme forces (SKM*).

The structure of the corresponding system of equations is similar to the structure of the system (54), except that the equations now contain both proton and neutron variables and, accordingly, there are now twice as many equations. By means of the approximate solution (see Sec. 3), the resonances can be readily classified by isospin. The isoscalar modes are shown in Fig. 9. It can be seen that allowance for the diffuseness of the edge of the nucleus (for the giant octupole resonance and 2^- level) and the realistic nature of the interaction lead to a significant improvement of the agreement between theory and experiment. The positions of the centroids of the low lying 3^- levels were determined from the formula

$$E^l_{3^-}=\sum_i E^{(i)}_{-3}B(E3,3^-_i\to gr)/\sum_i B(E3,3^-_i\to gr).$$

There is a rather large diversity in the data of different authors on the $B(E3)$ factors.[103–112] Therefore, in the figure we have given two experimental values—a maximal and a minimal one.

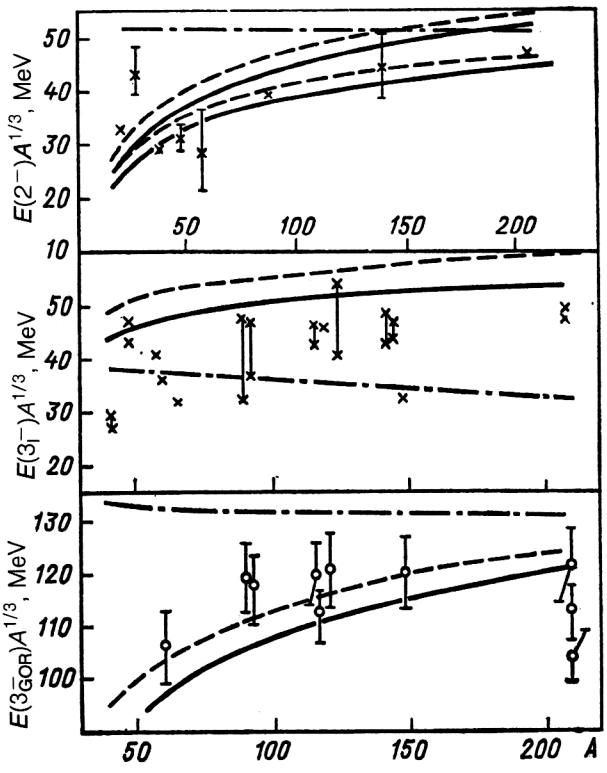

FIG. 9. Centroids of energies of the isoscalar giant octupole resonance (at the bottom), low-lying isoscalar 3^- states (in the middle), and the magnetic quadrupole resonance (isoscalar and isovector) (at the top). The calculations were made with the parameters of the equilibrium density from Ref. 78 (continuous curves) and Ref. 77 (broken curves). The chain curves are the result of calculation in the approximation of an incompressible nucleus with sharp edge. The experimental data are from Refs. 103–113.

The energies of the isovector 3^- excitations are shown in Fig. 10. Also plotted there are the first experimental data on the isovector giant octupole resonance.[79] It can be seen that the agreement between the theory and experiment is reasonable. By and large, the results of microscopic[68,114] and phenomenological[115,116] calculations of the isovector giant octupole resonance are fairly close to our results, but a different dependence on A is predicted. Our calculation can be approximated by the curve $E \simeq 51A^{-1/11}$ MeV, whereas phenomenology[68] gives $E \simeq 172A^{-1/3}$ MeV.

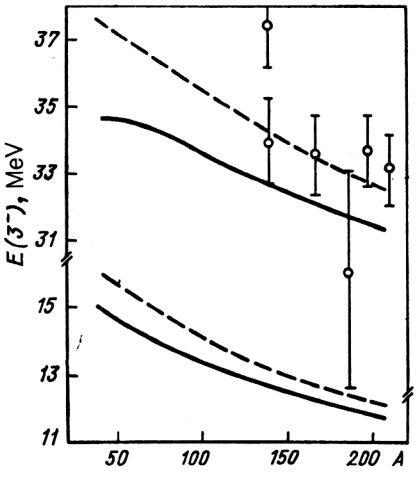

FIG. 10. Energies of isovector 3^- resonances. The parameters of the equilibrium density are taken from Ref. 78 (continuous curves) and Ref. 77 (broken curves). The experimental data are from Ref. 79.

5. VIRIAL EQUATIONS FOR TENSORS OF SECOND AND FOURTH RANK. 1^+, 2^+, 3^+, AND 4^+ EXCITATIONS IN THE APPROXIMATION OF AN INCOMPRESSIBLE NUCLEUS WITH SHARP EDGE

In this section, we take one further step in the direction of increasing the rank of the studied tensors. We shall investigate the dynamics of the tensors of fourth and second rank (simultaneously) in order to describe the giant hexadecapole resonance and the giant quadrupole resonance in the framework of a unified approach.[117]

Basic equations and approximations

In accordance with the rules formulated in Sec. 1 for the description of collective excitations of positive parity with multipolarity $\lambda \leqslant 4$, two systems of virial equations are required. The first system is provided by Eqs. (14)–(16), the second by Eqs. (24).

We assume that in the equilibrium state the nucleus has spherical shape, $u_i^{(0)} = 0, P_{ij}^{(0)} = \delta_{ij}P^{(0)}, P_{ijk}^{(0)} = 0$. We approximate the displacements ξ_i and the variations δP_{ij} and δP_{ijk} as follows:

$$\xi_i(\mathbf{r},t) = L_i(t) + \sum_{j=1}^{3} L_{i,j}(t)x_j + \sum_{j,k=1}^{3} L_{i,jk}(t)x_jx_k + \sum_{j,k,l=1}^{3} L_{i,jkl}(t)x_jx_kx_l; \tag{56}$$

$$\delta P_{ij}(\mathbf{r},t) = n(\mathbf{r},t)\left[D_{ij}(t) + \sum_{k=1}^{3} D_{ij}^{k}(t)x_k + \sum_{k,l=1}^{3} D_{ij}^{kl}(t)x_kx_l\right]; \tag{57}$$

$$\delta P_{ijk}(\mathbf{r},t) = n(\mathbf{r},t)\left[N_{ijk}(t) + \sum_{l=1}^{3} N_{ijk}^{l}(t)x_l\right]. \tag{58}$$

In the preceding sections, we have already given our arguments for such an approximation. In addition to them, we note that in the case of hexadecapole vibrations Eq. (56) is equivalent to Tassie's ansatz, and in the case of quadrupole vibrations is a generalization of it. In addition, for $\lambda = 4$ the expressions (56)–(58) are, in a certain sense, a "long-wavelength approximation," while for $\lambda = 2$ they represent an advance beyond it. The equilibrium values of the tensors P_{ij} and Π_{ijkl} and of the average field W are connected by relations that follow from the virial equations (15) and (24):

$$\int nx_j \frac{\partial W}{\partial x_i} d\mathbf{r} - \Pi_{ij} = 0;$$

$$\int nx_j x_k x_l \frac{\partial W}{\partial x_i} d\mathbf{r} - \Pi_{ij}^{kl} = 0;$$

$$\Pi_{ijkl} + \sum_{s=1}^{3} \int \frac{1}{\rho} \left[P_{ij} \frac{\partial P_{ks}}{\partial x_s} \right]_{ijk} x_l d\mathbf{r}$$

$$= -\frac{\hbar^2}{4m^2} \int nx_l \frac{\partial^3 W}{\partial x_i \partial x_j \partial x_k} d\mathbf{r}.$$

With allowance for the remarks and approximations listed above, the variations of the virial equations take the form

$$\ddot{V}_{i,j} - \delta\Pi_{ij} + \delta C_{ij} + 2\delta\sigma_{ij} - \delta U_{ij} = 0;$$

$$\delta\dot{\Pi}_{ij} + \frac{A\langle p^2 \rangle}{3m} \dot{L}_{ij} + A \frac{\langle p^2 r^2 \rangle}{3m} \dot{\tilde{L}}_{ij} = 0;$$

$$\ddot{V}_{i,jkl} + \delta C_{i,jkl} + \delta\Sigma_{i,jkl} - \delta\mathscr{U}_{i,jkl} - [\delta\Pi_{ij}^{kl}]_{jkl} = 0;$$

$$\delta\dot{\Pi}_{ij}^{kl} + 2A \frac{\langle p^2 r^4 \rangle}{15m} [\dot{L}_{i,jkl}]_{ij} + \delta_{kl} \frac{A}{9m} \left(\langle p^2 r^2 \rangle \dot{L}_{ij} + \frac{3}{5} \langle p^2 r^4 \rangle \dot{\tilde{L}}_{ij} \right) - \delta_{ij} \frac{A}{9m} \left(\langle p^2 r^2 \rangle \dot{L}_{kl} + \frac{3}{5} \langle p^2 r^4 \rangle \dot{\tilde{L}}_{kl} \right) - [\delta\Pi_{ijk}^{l}]_{kl} = 0; \quad (59)$$

$$\delta\dot{\Pi}_{ijkl} + \frac{A}{15m^3} [\delta_{jk}(\langle p^4 \rangle \dot{L}_{il} + \langle p^4 r^2 \rangle \dot{\tilde{L}}_{il})]_{ijkl} + \frac{\langle p^2 \rangle}{m^2 \langle r^2 \rangle} [\delta\Pi_{jkl}^{i}]_{ijkl} = 0;$$

$$\delta\dot{\Pi}_{ijk}^{l} + \frac{A}{3m^2} \left\{ \langle p^2 \rangle \delta_{il} D_{jk} + \frac{1}{3} \langle p^2 r^2 \rangle \left(2D_{jk}^{il} + \sum_{s=1}^{3} D_{jk}^{ss} \right) \right\} - \frac{1}{m} \delta_{jk} \int \frac{1}{n} x_l P^{(0)} \sum_{t=1}^{3} \frac{\partial}{\partial x_t} [n(D_{it} + \sum_{s,r=1}^{3} D_{it}^{sr} x_s x_r)] d\mathbf{r} - \frac{1}{m} \delta_{jk} \int \frac{1}{n^2} x_l \operatorname{div}(n\xi) \times P^{(0)} \frac{\partial P^{(0)}}{\partial x_i} d\mathbf{r} - \delta\Pi_{ijkl} - \chi_{ijk,l} = 0.$$

Here

$$\tilde{L}_{ij} = \sum_{s=1}^{3} (L_{i,jss} + L_{j,iss}),$$

$$\chi_{ijk,l} = \frac{\hbar^2}{4m^2} \delta \int nx_l \frac{\partial^3 W}{\partial x_i \partial x_j \partial x_k} d\mathbf{r}.$$

The tensors of the Coulomb forces have the form

$$C_{i,jkl} = \frac{1}{3} (2[C_{ij,kl}]_{jkl} + C_{ik,j,l} + C_{il,j,k} + C_{ij,k,l}),$$

$$2C_{ij,kl} = -\left(\frac{e_p Z}{A} \right)^2 \int \int n(\mathbf{r}) n(\mathbf{r}') x_k x_l \times \frac{(x_i - x_i')(x_j - x_j')}{|\mathbf{r} - \mathbf{r}'|^3} d\mathbf{r} d\mathbf{r}',$$

$$2C_{ij,k,l} = -\left(\frac{e_p Z}{A} \right)^2 \int \int n(\mathbf{r}) n(\mathbf{r}') x_k x_l' \times \frac{(x_i - x_i')(x_j - x_j')}{|\mathbf{r} - \mathbf{r}'|^3} d\mathbf{r} d\mathbf{r}'.$$

Expressions for their variations are given in Ref. 119. As usual, we decompose the tensors of the nuclear forces into surface and volume parts:

$$\int nx_j \frac{\partial U}{\partial x_i} d\mathbf{r} = 2\sigma_{ij} - U_{ij},$$

$$\int nx_j x_k x_l \frac{\partial U}{\partial x_i} d\mathbf{r} = \Sigma_{i,jkl} - \mathscr{U}_{i,jkl},$$

where

$$2\sigma_{ij} = -\int x_j U \frac{\partial n}{\partial x_i} d\mathbf{r},$$

$$U_{ij} = \delta_{ij} \int nU d\mathbf{r}, \quad \Sigma_{i,jkl} = 2[\sigma_{ij,kl}]_{jkl},$$

$$2\sigma_{ij,kl} = -\int x_j x_k x_l U \frac{\partial n}{\partial x_i} d\mathbf{r},$$

$$\mathscr{U}_{i,jkl} = [\delta_{ij} \mathscr{U}_{kl}]_{jkl}, \quad \mathscr{U}_{kl} = \int nU x_k x_l d\mathbf{r}.$$

We approximate the average field on the surface of the nucleus by the surface tension (38). Then for the tensors σ_{ij} and $\delta\sigma_{ij}$ we obtain the expressions given in Appendix 1, and for the tensor $\delta\Sigma_{i,jkl}$ and the quantum correction $\chi_{ijk,l}$ we have

$$\delta\Sigma_{i,jkl} = \frac{24\pi T}{mA} \left\{ V_{i,jkl} - \frac{R^2}{9} [\delta_{jk} V_{il}]_{jkl} \right\},$$

$$\chi_{ijk,l} = -\frac{5\pi T\hbar^2}{Am^3R^2} \left[\delta_{ij} \left\{ \frac{3}{5} V_{kl} - \frac{1}{R^2} \sum_{s=1}^{3} (V_{k,lss} + V_{l,kss}) \right\} \right]_{ijkl}.$$

Bearing in mind that the indices i, j, k, l take the values 1, 2, 3, we readily find that the system (59) contains 126 equations. They can be recombined in such a way as to obtain equations for modes of the definite multipolarities $\lambda = 4, 3, 2, 1, 0$.

Results of calculations

In the approximation of incompressibility, 0^+ levels do not appear. The theory gives two 4^+ levels with energies $E_4^{(1)} \simeq 70A^{-1/3}$ MeV and $E_4^{(2)} \simeq 212A^{-1/3}$ MeV. They are shown in Fig. 11 by the continuous curves (for nuclei out of the β-stability valley). Also plotted are experimental

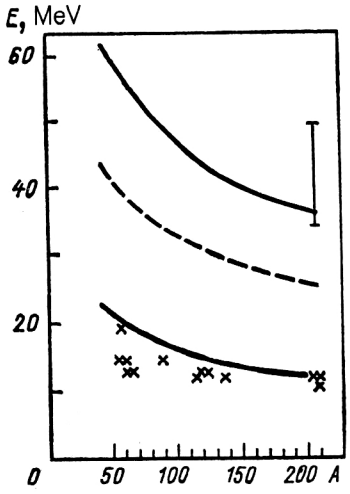

FIG. 11. Hexadecapole excitations. The continuous curves represent the exact calculation; the broken curve takes into account only the quadrupole deformation of the Fermi surface; the crosses are the experimental data from Refs. 98 and 120. The experimental bounds for the upper 4^+ excitation are from Ref. 98.

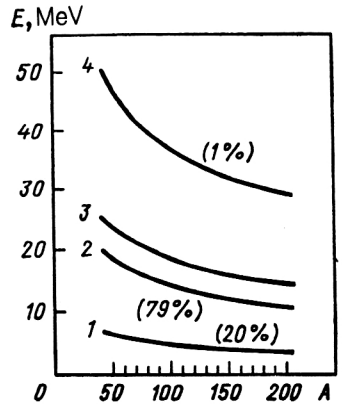

FIG. 12. Quadrupole excitations. The numbers in brackets give the percentage to which the isoscalar sum rule is exhausted.

data on the giant hexadecapole resonance from Refs. 98 and 120. It can be seen that the solution with lower energy agrees quite well with them. The discrepancy in the light nuclei must decrease when allowance is made for the diffuseness of the edge of the nucleus and for compressibility. This resonance corresponds approximately to the expected transition with $\Delta N = 2$ and is usually reproduced in RPA calculations of various degrees of complexity (Refs. 68, 114, 121, and 122). Without allowance for the Coulomb and surface forces, $E_4^{(1)}$ is lowered to $\sim 65A^{-1/3}$ MeV.

The second 4^+ resonance clearly lies above the expected excitation with $\Delta N = 4$, which, as a rule, is also obtained in RPA calculations.[68,114,121] The only exception is Ref. 122, which predicts a position of the second 4^+ level in the same region as we do. The one existing experimental indication[98] agrees with our calculation.

If we ignore the octupole and hexadecapole deformations of the Fermi surface (the variables $\delta\Pi^l_{ijk}$ and $\delta\Pi_{ijkl}$), then there remains only one solution (the broken curve), which has nothing in common with the experiment but does agree (if Coulomb and surface forces are also ignored) with the result of Ref. 12: $E_4 \simeq 150A^{-1/3}$ MeV.

The two levels make approximately equal contributions to the energy-weighted sum rule and completely exhaust it.

The system (59) gives one 3^+ level with energy

$$E_3 = 4\sqrt{\frac{2}{5}}\frac{v_F}{R} \simeq 116A^{-1/3} \text{ MeV}.$$

If the octupole deformation of the Fermi surface is ignored, the level is lowered to $\tilde{E}_3 = 3\sqrt{2/5}v_F/R = 87A^{-1/3}$ MeV. In Ref. 5, in which this deformation was not taken into account, it was obtained at approximately 103.5 MeV, i.e., approximately in the middle between E_3 and $\tilde{E}_3$. As yet there are no experimental data. From Eqs. (59) we obtain the four 2^+ levels shown in Fig. 12. They can be expressed approximately as $E_2^{(1)} \simeq 23.7A^{-1/3}$ MeV, $E_2^{(2)} \simeq 65.9A^{-1/3}$ MeV, $E_2^{(3)} \simeq 87.3A^{-1/3}$ MeV, $E_2^{(4)} \simeq 174.7A^{-1/3}$ MeV. The level $E_2^{(2)}$ is none other than the giant quadrupole resonance. We recall that in Secs. 2 and 3 it was calculated by means of the system of equations for only the second-rank tensors, and we then obtained $\tilde{E}_2^{(2)} \simeq 64.7A^{-1/3}$ MeV. We see that the extension of the computational scheme to the fourth-rank tensors has had practically no influence on the energy of the giant quadrupole resonance, and this indicates stability of the method. The lowest level ($E_2^{(1)}$) exhausts ~20% of the energy-weighted sum rule, in agreement with the results of Ref. 123, while the giant quadrupole resonance ($E_2^{(2)}$) exhausts about 79%, leaving only 1% of the sum rule for the fraction of the two higher 2^+ excitations. When they are all taken together, they completely exhaust the sum rule. From this it may be concluded that the lowest-lying 2^+ excitation is the centroid of all the 2^+ states of the nucleus lying below the giant quadrupole resonance. The experimental data of Refs. 124–130 do not contradict such an interpretation. It should be emphasized that for our method the fact that a low-lying 2^+ state appears of basic importance, since it can be reproduced in microscopic calculations[114,122,131] and is not obtained in the phase-space method.[39,132,133]

It appears that we can relate the nature of at least one of the two high-lying 2^+ excitations to the dynamics of the toroidal quadrupole moment:[99]

$$T_{ij} = \frac{e_p Z}{28cA}\int n[4x_i x_j(\mathbf{r}\cdot\mathbf{u}) - 5r^2(x_i u_j + x_j u_i) + 2r^2(\mathbf{r}\cdot\mathbf{u})\delta_{ij}]d\mathbf{r},$$

since its variation is a linear combination of the variables represented in the system (59):

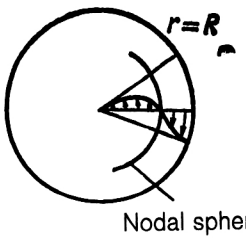

FIG. 13. Field of collective velocities corresponding to 1^+ excitation.

$$\delta T_{ij}=\frac{e_p Z R^2}{28mcA}\left\{2\dot{V}_{ij}-\frac{7}{R^2}\sum_{s=1}^{3}[\dot{V}_{i,jss}]_{ij}+\frac{2}{R^2}\delta_{ij}\sum_{s,t}\dot{V}_{s,stt}\right\}.$$

This assumption is supported by the simplified RPA calculation of Refs. 100 and 134, which predicts a 2^+ level of vortex nature at energy $\sim 93A^{-1/3}$ MeV, which is very close to our result for $E_2^{(3)}$. Also predicted there is a low-lying "toroidal" 2^+ level, and therefore it cannot be ruled out that our level at $23.7A^{-1/3}$ MeV has a significant admixture of a vortex component.

The system (59) also gives one 1^+ level with energy

$$E_1=132.6A^{-1/3}\ \text{MeV}.$$

Neglect of the octupole deformation of the Fermi surface raises it to $E_1 = 162.4A^{-1/3}$ MeV, in almost exact agreement with the result of Ref. 5 ($\sim 161A^{-1/3}$ MeV), in which the octupole deformation of the Fermi surface was also not taken into account. In hydrodynamics, such a mode has long been known.[5] The distribution of currents corresponding to it is shown in Fig. 13, which is taken from Ref. 9.

To complete this section, we note that the influence of the quantum correction on all four modes is negligibly small.

CONCLUSIONS

We briefly summarize the main results and conclusions. Beginning with the equation for the density matrix in the framework of the time-dependent Hartree–Fock method, we have obtained a system of equations, first for the moments of the Wigner function in the momentum space, and, after this, for the moments in the complete phase space. Such moments of the Wigner function play the part of collective variables. The Hartree–Fock theory is reduced to a form that resembles hydrodynamics, namely, the dynamics of a Fermi liquid.

As one of the most important points we have considered rules for closing the hierarchy of equations for the moments. It follows from these rules that for correct description of the dynamics of a mass (or charge) tensor of rank k ($J_{i_1\ldots i_k} = \int \rho x_{i_1}\ldots x_{i_k} d\mathbf{r}$) it is necessary to take into account the deformation of the Fermi surface described by the tensors

$$P_{i_1\ldots i_l}=m^{1-l}\int (\mathbf{p}-m\mathbf{u})_{i_1}\ldots(\mathbf{p}-m\mathbf{u})_{i_l} f d\mathbf{p}$$

of all ranks up to $l = k$.

The method has been used to study the shapes of rotating nuclei and quadrupole and monopole excitations in them (Refs. 13–15, 26, and 74), and also to study collective states of negative parity in spherical nuclei (Refs. 27, 28, 32, and 33): octupole, dipole, and magnetic quadrupole excitations. The calculations have been made both in the approximation of a sharp edge of the nucleus with surface tension and for nuclei with a diffuse edge and realistic interaction (SKM*).

In the approximation of a sharp edge in spherical nuclei, 4^+, 3^+, and 1^+ excitations have been investigated.[117] The excitation energies of the collective states and their electromagnetic properties have been considered. The possibility of describing the widths of the resonances in the framework of the approach has been demonstrated.[64]

Our results speak quite persuasively for the advantages of the method of moments. Indeed, in the framework of a unified approach it is possible to describe the static and dynamical properties of the nuclei and the isoscalar and isovector excitations; besides giant resonances, low-lying modes are obtained. Despite the rather complicated realistic interaction, the computational work is comparatively simple, namely, one finds roots of polynomials whose coefficients are ordinary one-dimensional integrals. The nonlocality of the interaction does not lead to any additional difficulties. A very interesting fact is the appearance of low-lying 3^- and 2^+ levels; this fact offers hope of a more detailed description of the low-lying part of the spectrum as the number of degrees of freedom is increased (as study is extended to the dynamics of the tensors of higher and higher rank).

The theory described in the review, although relatively simple, is capable of reproducing the known experimental data in all cases in which such comparison is possible, and it also gives estimates similar to the results of other more laborious theoretical calculations.

In the approach, definite assumptions are made about the velocity field of the nuclear matter. They may lead to certain errors in the estimate of physical observables. To correct them, it is necessary, in accordance with the theory, to take into account the tensors $J_{i_1\ldots i_k}$ of higher rank. The difficulties that arise in this way are more than compensated by the possibility of avoiding the difficulties of another kind encountered in the solution of the differential equations for $\rho(\mathbf{r},t)$, $\mathbf{u}(\mathbf{r},t)$, $P_{ij}(\mathbf{r},t)$, etc. Namely, to solve a differential equation, it is necessary to specify boundary conditions on the surface of the nucleus. They are fairly obvious on the "outer" surface, where there is no matter. But these conditions are of no practical use until one has found a method of describing the dynamics of the surface layer; this, for example, is the aim of Refs. 135 and 136.

In the review we have shown that it is possible to give a quantitative description of the properties of nuclear vibrational states using parameters of the liquid-drop model, i.e., a model intended to describe the static properties of

nuclei. Thus, in accordance with the results of Sec. 4, it is the surface tension and fissility parameters of the drop of Fermi liquid that determine the position of the 3^- excitations and the strength of the $E3$ transitions in a region of energies of the order of one main shell. There is hope that the proposed method will help to fill the gap between the macroscopic (liquid drop) and microscopic (shell model, etc.) approaches to nuclear dynamics. Although the conceptual[137] relationship between these approaches is well understood, further study is required.

We outline the possible further applications of the method. It is obvious that the inclusion of spin degrees of freedom should not lead to serious difficulties—it is simply necessary to double the number of all possible equations of motion. It would also be interesting to take into account elements of the density matrix nondiagonal with respect to the spin and isospin, since this would make it possible to describe spin-flip and charge-exchange processes. The problem of describing large-amplitude motions, in particular fission, is also not without hope. To describe the rotation of nuclei possessing static deformation, it is necessary to work with Riemann ellipsoids,[16] i.e., with objects possessing a nonzero velocity distribution $\mathbf{u}^{(0)}(\mathbf{r})$ in the state of secular equilibrium. The description of reactions without contact, of the type of Coulomb excitation, should obviously not encounter any problems. Some considerations with regard to contact reactions were advanced in Ref. 138. Calculations of the dynamics of the tensors of the fifth, sixth, etc., rank (especially with realistic interaction) are held back for the moment by the laboriousness of the calculations, but with the development of the possibilities of analytic programming here too great advance can be expected.

APPENDIX 1. TENSORS OF THE SURFACE AND COULOMB ENERGY IN THE APPROXIMATION OF A SHARP EDGE OF THE NUCLEUS AND THEIR VARIATION

The tensors of the surface and Coulomb energy in the approximation of a sharp edge of the nucleus and their variations are

$$C_{ii}=bXA_f a_f^2/r^2;\quad \sigma_{ii}=(\mathscr{A}_f+\mathscr{A}_k)b/4r_0^2\quad (i\neq j\neq k);$$

$$\delta C_{ij}=X\Omega_0^2\left[2B_{ij}V_{ij}+\delta_{ij}\sum_{k=1}^{3}V_{kk}(B_{ik}-A_k)\right],$$

$$\delta\sigma_{ij}=\frac{1}{4}\Omega_0^2\left[2\mathscr{A}_{ij}V_{ij}-\delta_{ij}\sum_{k=1}^{3}V_{kk}(\mathscr{A}_k+\mathscr{B}_{ik})/a_k^2\right].$$

Here $\Omega_0^2=5b/2mAr_0^2$, $\mathscr{A}_{ij...}$, $\mathscr{B}_{ij...}$ and $A_{ij...}$, $B_{ij...}$ are the so-called multiple-index symbols introduced in Refs. 16 and 49:

$$\mathscr{A}_{ij...}=\int_0^\infty\frac{R^6dt}{\Delta_R(a_i^2+t^2)(a_j^2+t^2)...},$$

$$A_{ij...}=\int_0^\infty\frac{R^3dt}{\Delta_c(a_i^2+t)(a_j^2+t)...};$$

$$\mathscr{B}_{ij...}=\int_0^\infty\frac{R^6t^2dt}{\Delta_R(a_i^2+t^2)(a_j^2+t^2)...},$$

$$B_{ij...}=\int_0^\infty\frac{R^3tdt}{\Delta_c(a_i^2+t)(a_j^2+t)...};$$

$$\Delta_R^2=(a_1^2+t^2)(a_2^2+t^2)(a_3^2+t^2),$$

$$\Delta_c^2=(a_1^2+t)(a_2^2+t)(a_3^2+t).$$

APPENDIX 2. CONNECTION BETWEEN THE COMPONENTS OF THE IRREDUCIBLE AND CARTESIAN TENSORS

The relations between the components of the Cartesian tensor $V_{i,j}$ and the variations of the quadrupole moment $Q_{2\mu}$, angular momentum I, and mean-square radius $\langle r^2\rangle$ are:

$$\delta Q_{20}=-\chi(V_{11}+V_{22}-2V_{33});$$

$$\delta Q_{2,\pm1}=\mp\chi\sqrt{6}(V_{13}\pm iV_{23});$$

$$\delta Q_{2\pm2}=\chi\sqrt{3/2}(V_{11}-V_{22}\pm 2iV_{12});$$

$$\chi=\sqrt{5/\pi}Ze_p/4mA;$$

$$\delta I_1=\dot{V}_{3,2}-\dot{V}_{2,3}-\Omega V_{31};$$

$$\delta I_2=\dot{V}_{1,3}-\dot{V}_{3,1}-\Omega V_{32};$$

$$\delta I_3=\dot{V}_{2,1}-\dot{V}_{1,2}+\Omega(V_{11}+V_{22});$$

$$\delta\langle r^2\rangle=\frac{1}{m}\sum_{i=1}^{3}V_{ii}.$$

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